ICME-13 Monographs

Judit N. Moschkovich
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Marcus Schütte *Editors*

Language and Communication in Mathematics Education

International Perspectives

ICME13
Hamburg 2016

Springer

ICME-13 Monographs

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ISSN 2520-8322 ISSN 2520-8330 (electronic)
ICME-13 Monographs
ISBN 978-3-319-75054-5 ISBN 978-3-319-75055-2 (eBook)
https://doi.org/10.1007/978-3-319-75055-2

Library of Congress Control Number: 2018930111

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Printed on acid-free paper

This Springer imprint is published by the registered company Springer International Publishing AG part of Springer Nature
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

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Part I
Overviews of Language and Communication Research in Mathematics Education

International Perspectives on Language and Communication in Mathematics Education

David Wagner and Judit N. Moschkovich

Abstract This chapter introduces this volume, which arose from the conversations among 90 scholars from 23 countries within the topic study group on Language and Communication in Mathematics Education at the 16th International Congress of Mathematical Education, which convened in Hamburg, Germany. The chapter describes the goals of the topic study group and the diversity of contributions, and it introduces the papers that were selected for elaboration and publication in this volume.

1 Background

Language and communication are recognized to be core components in the teaching and learning of mathematics, but there are many outstanding questions about the nature of interrelationships among language, mathematics, teaching, and learning. Recent research has demonstrated the wide range of theoretical and methodological resources that can contribute to this area of study, including those drawing from cross-disciplinary perspectives influenced by, among others, sociology, psychology, linguistics, and semiotics.

Thus it is warranted for scholars who take these questions seriously to gather and discuss their latest work. An important context for this kind of knowledge sharing is presented every four years at the *International Congress of Mathematical Education* (ICME). This is a large conference that convenes topic study groups for important research foci within the larger field. One of these topic study groups is entitled Language and Communication in Mathematics Education. At the ICME 13 conference, convened in Hamburg, Germany in 2016, this topic study group (TSG) was organized by co-chairs Judit N. Moschkovich (USA) and David Wagner

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© Springer International Publishing AG, part of Springer Nature 2018
J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_1

(Canada), and their team including Arindam Bose (India and South Africa), Jackeline Rodrigues Mendes (Brazil), and Marcus Schütte (Germany).

This volume arose from the conversations among 90 scholars within the topic study group (TSG) on Language and Communication in Mathematics Education at the 16th ICME conference. There are other regular gatherings of scholars who study language and communication in the field, but ICME is special because it draws the most diverse group in terms of countries represented. At ICME 16, the TSG included scholars from the following countries: USA (14), Germany (13), Italy (8), China (7), Japan (7), South Africa (6), Canada (4), Korea (4), Norway (3), Turkey (3), United Kingdom (3), Brazil (2), Colombia (2), Denmark (2), Ireland (2), Mexico (2), Spain (2), Sweden (2), Thailand (2), Algeria (1), Greece (1), India (1), Uruguay (1). While some of the participants are from countries typically represented at international conferences, it is clear that this TSG includes scholars from regions that are relatively new to research on the topic of language and communication in mathematics education. We structured the TSG sessions to feature and represent the most innovative approaches to the research and also to give us insight into the concerns brought from regions that are relatively new to the conversation. This volume comprises elaborated versions of just some of the papers presented in the TSG during the conference, benefitting from time to extend the writing based on the feedback received at the conference. Some of the chapters in this volume are the first English language publications for the authors beyond conference proceedings. This speaks to the importance of gatherings that assemble scholars representing such diversity.

The TSG invited presentation, discussion, and reflection on current approaches to research on language and communication related to learning and teaching mathematics. The invitation to contribute to the TSG described "language and communication" in its broadest sense to mean the multimodal and multi-semiotic nature of mathematical activity and communication, using not only language but also other sign systems. The TSG thus welcomed contributions focusing on all modes of communication—oral, written, gestural, visual, etc. The TSG built on the strong body of research in mathematics education that addresses these issues and also considered important questions that remain.

Several themes described in the TSG-31 description were addressed during the main sessions: the role of theory in understanding language and communication in mathematics education; multiple methods for researching mathematics education; relationships among language (and other sign systems), mathematical thinking, and learning mathematics; language, communication, and mathematics in classrooms and communities; and using theoretical and methodological tools from other disciplines such as linguistics, semiotics, discourse theory, sociology, etc.

2 Language and Communication in Relation to the Field

One of the challenges in organizing a topic study group on language and communication is that mathematics education research can be connected to the themes in more than one topic study group. As noted in the chapter by David Pimm in this volume, research in the field often features data in the form of language excerpts from books or from transcriptions of oral classroom dialogue. Likewise, attention to language typically is situated in a context that connects with the themes of one of the other TSGs. Thus, researchers need to decide which conversation they want to be part of in any given ICME conference. The TSG entitled “Mathematics Education in a Multilingual and Multicultural Environment” (TSG-32), in particular, necessitated a choice for scholars who focus on multilingual contexts because their research fits very well in either that TSG or the Language and Communication TSG (TSG-31).

For this reason, our TSG leadership team collaborated with the leaders of TSG-32 to convene a joint session at ICME. TSG-32 was co-chaired by Richard Barwell (Canada) and Anjum Halai (Pakistan), with team members Guida de Abreu (UK), Aldo Parra (Colombia), and Lena Wessel (Germany). The joint session of TSGs 31 and 32 provided the opportunity for participants in the two TSGs to discuss common concerns and significant distinctions in mathematics education research on language that considers (or not) multi-lingual and multi-cultural dimensions. The session comprised a panel and discussion focused on the theme: “Intersections and differences in work on language in monolingual and multilingual/multicultural classrooms and settings”. The panelists were Richard Barwell, Arindam Bose, Aldo Parra, Jackeline Rodrigues Mendes, David Wagner and Lena Wessel. The panel was chaired by Judit N. Moschkovich and Marcus Schütte. As a prompt for the discussion, the panelists provided a handout of some “provocative statements” related to the TSG foci and participants were invited to discuss the following questions:

- What do you see or experience as points of intersection between these two foci: mono- and multilingual/multicultural?
- What do you see or experience as differences between these two foci: mono- and multilingual/multicultural?
- Why do you think these two topics are treated as separate?
- How can insights from one focus contribute to the other focus and vice versa?

A productive discussion of these questions involving panel members and the audience then ensued.

Here is a list of the statements intended to be provocative, not necessarily statements that each panellist held to be true.

> Aldo Parra: Discussions on language and multilingualism in mathematics education are being done mainly on the technical aspects, in order to achieve expertise on the mathematical content, without considering cultural values and practices, nor problematizing mathematics as cultural and historical production. This could subsume the field production

to “the banality of expertise”—i.e., a renewed type of deficitarian approach. This is one reason why multicultural perspectives can contribute to mono- and multilingual foci.

Arindam Bose: Politics of social goods (gender, race, class, power, status, etc.) are dependent on and build language communication, and therefore affect mathematics learning and teaching.

David Wagner: Various foci warrant attention in mathematics classroom discourse (not only multi-lingual aspects)—for example, context-specific power relationships and authority, which are structured and sustained by language practices—but any of these foci are informed by attention to linguistic and cultural diversity at play.

Jackeline Rodrigues Mendes: An intersection between these two foci can be the central role of language in different mathematical teaching and learning processes in mono- and bi-multilingual/multicultural scenarios—language, thought not only in a verbal sense (with emphasis on differences between speaking and writing, linguistic characteristics of these processes, interactional discourse, etc.), but including other forms of making sense like visual language, gestural language, interactional silence and other cultural symbolic systems as modes of communication. In this way, if we consider sociocultural issues that support processes of meaning and sense making through language, we can think language as a social practice.

Lena Wessel: Multilingual students are socially disadvantaged and only have limited proficiency in the language of instruction, so they are also low achievers in mathematics and need additional support which monolinguals don’t need.

Richard Barwell: It is more helpful to think of a continuum between multilingualism/multiculturalism and an idealised state of monolingualism/monoculturalism. Research that treats mathematics classrooms as monolingual/monocultural risks perpetuating the idea that monolingualism/monoculturalism is the norm, when in fact most students and teachers of mathematics are multilingual/multicultural, hence contributing to the marginalisation of many students and teachers.

3 This Volume

This volume is organized with the following four themes. The first section includes chapters that address the scope of research on language and communication in mathematics education research. The section begins after this introduction with David Pimm’s reflections on the centrality of language to the whole field of mathematics education research. Next, Marcus Schütte (whose name is sometimes spelled Schuette) writes from a perspective that is not English language dominant, and describes developments in language and communication in the field, particularly from research outside of English dominant regions. The section closes with Judit N. Moschkovich’s reflections on her years of research (and reading the research of others) to develop recommendations for conducting mathematics education research with a focus on language and communication.

We note that there have been a number of recent publications that give excellent overviews of language and communication research in mathematics education. We highlight a few of these here. In 2014, a special issue of *ZDM: The International Journal of Mathematics Education* compiled recent work on this topic.

The co-editors, Morgan, Craig, Schütte, and Wagner (2014) contributed an article that described various streams of research within language and communication in the field, namely: analysis of the development of students' mathematical knowledge, understanding the shaping of mathematical activity, understanding processes of teaching and learning in relation to other social interactions, and multilingual contexts.

In the *First Compendium for Research in Mathematics Education,* recently published by the National Council of Teachers of Mathematics, there are two chapters that review research literature relating to language and communication. One is a critical analysis of mathematics classroom discourse literature written by Herbel-Eisenmann, Meaney, Bishop Pierson, and Heyd-Metzuyanim (2017). The other reviews the research on language diversity including contexts of second language, and involving bilingual and multilingual learners, written by Barwell, Moschkovich, and Setati Phakeng (2017).

The second section in this volume collects research that focuses on learners in mathematics classrooms. First Jenni Ingram, Nick Andrews and Andrea Pitt use a conversation analytic approach to explore the interactional structures that make student explanations relevant. For this research set in secondary mathematics classrooms in the United Kingdom, they are most interested in students' interpretations of the interaction as requiring an explanation and constraining the type of explanation. Second, Benadette Aineamani's research is in the context of a grade 11 classroom in a township school in South Africa. She considers the way textbook activities and teacher questions prompt student reasoning. Third, Carina Zindel analyses students' solution processes for function word problems focused on the intertwined conceptual and language demands of function word problems. The 16 design experiments in a laboratory setting with ninth and tenth graders highlights the need for adequately connecting verbal and symbolic representations. Fourth, Marei Fetzer and Kerstin Tiedemann develop a theoretical framework that allows a new perspective on the interplay of language and objects in the process of abstracting. Their work includes a brief example from their German context. Fifth, David Wagner and Annica Andersson, problematize the identification of an interaction as a mathematical situation by pointing out multiple intersecting discourses. This research uses an interaction between a researcher and some 4-year-old Canadian children. Finally, Kirstin Erath introduces a framework for analysing collective explanations in whole-class discussions—a conceptualisation of explaining as mathematical discursive practices of navigating through different epistemic fields. This links theories from linguistics and mathematics education. and simultaneously highlights the interactive nature of explaining processes while keeping the mathematical content in focus.

The third section in this volume collects research that focuses more on mathematics teachers. It begins with Judith Jung's exploration of the potential for social and content-related participation in inclusive school teaching. Using transcripts of video-recorded lessons in a Year 1 class analysed through interactional analysis, the chapter describes how interaction was strongly structured by the teacher through repetitive sequences, enabling the participation of many pupils but, at the same time, providing them few opportunities to participate outside of these structures.

Second, Kaouthar Boukafri, Marta Civil and Núria Planas, working in the context of 12-year-old students studying geometry in Catalonia, Spain, explore how a teacher's use of revoicing promotes students' mathematical thinking and learning opportunities. Third, Konstantinos Tatsis and David Wagner juxtapose two analyses of episodes from a year 9 mathematics classroom in Canada—an analysis based on politeness theory, and an analysis based on an authority framework. Fourth, Lorena Trejo-Guerrero and Marta Elena Valdemoros-Álvarez report the results of research on the construction of meaning for natural numbers, through the development of division in a primary classroom in Mexico. The analysis of the case study of one teacher focuses on the significance of mathematical language in one lesson that contrasted two different approaches to division with natural numbers, canonical division and Egyptian division. Finally, Raquel Milani develops the idea of dialogue in the context of her work with pre-service teachers in Brazil. She describes how she promoted the development of a concept of dialogue and how one of the prospective teachers did this with her students.

The final section in this volume collects research that focuses on aspects relating to their multilingual contexts. First, Alejandra Sorto, Aaron T. Wilson and Alexander White, working in a USA context, show correlations between teachers' mathematical knowledge and knowledge of teaching linguistically diverse learners, and show that these were strongly associated with rich mathematics and attention to students as learners during instruction. Second, Christine Bescherer and Pelagia Papadopoulou-Tzaki study the change of language awareness as teacher students create audio podcasts on a mathematical topic. This is set in a context of remedial mathematics instruction for students learning German. Third, Lindiwe Tshabalala explored how a grade 7 teacher promoted mathematical reasoning in a multilingual mathematics class of English second language learners in a school in an informal settlement West of Johannesburg South Africa. The analysis showed that the teacher focused on developing learners' procedural fluency and that this focus was accompanied by the dominant use of English by the learners. Finally, Jackeline Rodrigues Mendes describes ethnographic research carried out in a multilingual context in the Xingu Indigenous Park in Brazil, a setting of indigenous teachers' education. The chapter examines the process of developing a mathematics textbook written in indigenous language by Kaiabi teachers to be used in indigenous schools in the park. This discussion explores the Kaiabi cultural meaning of 'number' in community practices and the practices arising from contact with non-indigenous society.

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Sixty Years (or so) of Language Data in Mathematics Education

David Pimm

Abstract This chapter, based both on pre-ICME-13 conference documents as well as on the author's actual panel presentation made at TSG 31, covers a range of themes concerned with the issues of 'language data' in mathematics education. It also addresses several instances from its history, including word problems, classroom language and transcription, in addition to the mathematics register, its syntax, semantics and pragmatics.

1 Introduction

My title contains a conscious hedge. While it is likely not a span of *exactly* sixty years that language data have been offered as central and focal for research efforts in mathematics education, it seems close enough as a temporal marker to consider what sorts of language data have counted and how they have been dealt with over such a period of time. But it also raises for me some interesting ('Patient zero') thoughts in terms of what the first reporting and analysing of such data in mathematics education was like (for what purpose, in what form, what was done with it, to what end and why). And, also, whether we, some sixty (or so) years later, might recognise and still acknowledge it as language data research, not to mention moving on to the ever-pressing question of what mathematics education language-data-based research might come to look like in the future. Here are a few possibilities from the past.

- Is it to be found in research work on arithmetic word problems, which was a popular US doctoral research topic in the 1960s, but also discussed by Thorndike (1922)?

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© Springer International Publishing AG, part of Springer Nature 2018
J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_2

- Was it in an attempt to depict an element or aspect of a mathematics classroom setting or interaction (and almost certainly fragments of such accounts appeared in professional publications and journals prior to research ones)?
- Was it actually to be found in the research foci and accounts of psychologists who regularly, even persistently, decided that instances of mathematical thought offered the best example of 'pure' cognition to be found, so used mathematical prompts or probes in order to find out, not about mathematical thinking per se, but about 'thinking' *tout court*? And the common way they attempted to access thought, mathematical or otherwise, was through speech.

Mathematics education, even to this day, has not fully emerged from the shadow of academic psychology of a certain sort[1]—one surface trace of this can be found in the persistently lingering 'P' in the name of the PME annual conference, despite a likely majority of the papers presented there in any given year having very little, if any, connection to that field. Instead, papers regularly draw on other cognate disciplines, including sociology, socio-linguistics and linguistics (though even to this day it is not straightforward to have a solely linguistics-rooted mathematics education paper accepted by a mainstream mathematics education journal).[2] (For more on the 'turn to language', see Morgan, 2006.)

A further echo exists in the broad (and unquestioned) adoption of the genre and style (including section order and dull general headings) of academic psychology writing by most mathematics education journals. One consequent aspect of this is the overvalued mirroring of the quasi-scientific basis for a certain sort of empirical psychology study, one based on isolated experiments and statistical analysis, rather than, say, a classroom-based study of classroom phenomena (thus, the classroom as both setting and site for much mathematics education research). There is an interesting question as to whether the emergence of language as data relates more than simply chronologically to the field's move away from psychological approaches: in particular, did attention to language as data prompt a move away from psychology (or, indeed, was it vice versa)?

I recently came across Richard Nisbett and Timothy Wilson's (1977) piece entitled 'Telling more than we can know'.[3] This article addresses, among several other things, self-reports of action rationales (in response to 'how' and 'why' questions about

[1]For a previous ICME, held in Québec in 1992, I prepared a talk (which I was unable to give) entitled 'Another psychology of mathematics education' (see Pimm, 1994).

[2]The same is true, interestingly, of papers on mathematics education and technology (see Sinclair, 2017). However, while there are general technology and education journals (as there are for linguistics and education), there are not any specialist language and mathematics education journals comparable with *Digital Experiences in Mathematics Education* or STEMmy journals such as *The Canadian Journal of Science, Mathematics and Technology Education*.

[3]This article title contains a very nice play on the wording of Michael Polanyi's (1966/2009) startling claim-as-fact, in his book *The Tacit Dimension*, that, '*we can know more than we can tell*' (p. 4; *italics in original*). Curiously, the header throughout Nisbett and Wilson's article is the subtitle, 'verbal reports on mental processes', which contains more characters than the actual main title.

behaviour, for example)—a common-enough twentieth-century research practice in social psychology—coming into collision in the US in the 1960s with some other psychologists (such as George Miller or Ulric Neisser). The latter doubted we can have any direct or worthwhile access to high-level mental processes, "such as those involved in evaluation, judgment, problem solving, and the initiation of behavior" (p. 232), merely to their products. And this not to mention on the other side of things Sigmund Freud and his influence on the question, which includes the prospect that we actually do not or cannot awarely 'know' certain things, because we repress them.

Nisbett and Wilson note the widespread ease of interviewees' responses to such questions, and neatly turn the tables on the challengers by inviting them to account for such responses. Nevertheless, the article provides a consistent attempt to undermine and devalue self-report data. This widely cited article from the 1970s reflects a second-wave, anti-introspection challenge, perhaps echoing an earlier generation's attempts to require a focus on objectively observable data as the only data worth having (see Christopher Green's magnificent 1992 trace history of physicist Percy Bridgman's 'operation[al]ism' in psychology).

Another contested practice involved in generating language data is that of the 'clinical' interview. Two of its staunch defenders in relation to mathematics (and science) education have been Herbert Ginsburg (e.g. his 1981 article) and Andrea diSessa (e.g. his 2007 one), both writing about a language data source once again under challenge from psychologists of a certain stripe, albeit some twenty-five years apart. And even Ginsburg's actual title frames it as 'psychological research on mathematical thinking'.

In both of these examples, it is not so much the method of data generation that was under attack as the value and validity of the speech data generated that was/is being called into question. Behind this seemingly gentle enquiry is the question of when, where and how mathematical language itself, in some written, spoken or perhaps signed form, was made into an *object* of study, into a thing in its own right, rather than simply being seen as a (transparent) carrier of something else (meaning, significance, thought, emotion, …)? And all the transcription practices (I discuss this a little further below) make one realise that, whatever 'it' is, you can never fully capture it. Speech (as well as para-linguistic phenomena) is profoundly different from writing. I have offered these micro-synopses here to remind us all that it has not always been plain sailing in mathematics education with regard to the presentation and analysis of language data. There is a much deeper intellectual history of this area still waiting to be written.

2 Word Problems

Word problems form a stable linguistic target with a very long history, at least as a mathematical–pedagogical object (e.g. some problems in the Rhind Mathematical Papyrus, a text, claiming to be a copy of an earlier text, dating from nearly 2000 BCE). The very name 'word problem' seems to identify it by its linguistic aspects.

They are also primarily written, usually in 'textbooks', hence already a textual object of sorts. This feature is something which has made them a less transient phenomenon, even if they may frequently be read aloud in a classroom setting. (For a single instance of this, see, for example, Herbel-Eisenmann & Pimm, 2014.)

As I mentioned in passing above, they became subject to a strong doctoral study focus in the US in the 1960s (where I encountered them *en bloc* as a doctoral student myself at the University of Wisconsin-Madison in 1977–9, charged with producing a research review in relation to addition and subtraction word problems, work which in a small way fed into Carpenter, Moser, and Romberg 1982). Many of the doctoral studies that I read focused on aspects of syntax (occasionally semantics), features which were used to attribute an *a priori*, variably weighted, numerical degree of difficulty (based on a range of linguistic features such as active or passive voice and other measures of syntactic complexity, the number of words in the problem or per sentence, …) to arithmetic word problems, a purported 'measure' which was then compared empirically with students' success with them (a different notion of 'actual' difficulty). An example of such an approach to arithmetic word problems can be seen in Jerman and Rees (1972), while an instance in regard to algebraic ones can be seen in Lepik (1990). Jerman and Rees claim, "A basic assumption of this approach is that the structure of the arithmetic problem itself, to a large measure[,] determines its difficulty level" (p. 306).

These early instances of (basic) linguistic notions used to engage with a phenomenon of mathematics education also reflect a style and genre of research in which raw data was immediately quantified and then discarded and the numerical measures then 'became' the data that was (parametric-statistically) analysed. The more contemporary style tends not to turn words into numbers (something which, for me, echoed the ancient practice of *gematria*), preferring to work with them more on their own terms.

In the 1970s, Pearla Nesher undertook a considerable amount of research in this area (e.g. 1972; Nesher & Teubal, 1975) and she was a central figure in broadening the focus of attention from lexico-grammmatical features to questions of semantics (e.g. Nesher & Katriel, 1977). However, one significant more recent example (involving both syntax and semantic elements) can be found in Susan Gerofsky's extensive and sophisticated work (e.g. 1996, 2004) on word problems, not least where she draws on Levinson's (1983) account of pragmatics (and of L-tense and M-tense of verbs in particular, in relation to time coding; see his chapter on deixis, pp. 54–95).

Linguistic verb tense (L-tense) refers to conventional grammatical tense in a language, while meta-linguistic verb tense (M-tense) signals a deictic category that encodes an event relative to the coding time (CT) of an utterance. Gerofsky (1996) writes:

> In an M-tense system, we distinguish the temporal location of events in relation to CT: *past* refers to events prior to CT, *present* to events spanning CT, *future* to events succeeding CT, *pluperfect* to events prior to *past* events (which are themselves prior to CT) and so on. […] I have found that determining M-tense in mathematical word problems is problematic. (p. 40)

As an illustrative instance, Phillips (2002) provides discussion of a grade-four, student-generated word problem:

> Jane and Lucy both weigh 35 kilograms. Lucy went on a diet and now she is 30 kilograms and Jane has gained 7 kilograms. How much more does Jane weigh than Lucy? (p. 254)

Attending to the verbs and their L- and M-tenses (as well as the temporal deixis of 'now' as marking the CT of the problem) reveals the students' apparent unawareness of this feature of word problems.

The research topic of mathematics word problems and their characteristics has not gone away. Research attention is still in place, while certain analyses have gained in linguistic sophistication. (For an encyclopædia entry on research on word problems in mathematics education, see Verschaffel, Depaepe, & van Dooren, 2014, although there is no specific mention of linguistic analysis of word problems either in their keywords or in the body of their text.)

3 Classroom or Research Events and Their Records: Making a Thing of Things

Earlier, I made mention of the need to make an object out of language in order to study it. In this instance, language is no different from anything else: all events take place in time and vanish. Records are required in order to make time stand still (even if only temporarily), as well as to allow repeated entry over time, as far as is possible, via the record to the event itself. The records become proxies for the events themselves. (For a little more on this, see Pimm, 2018.)

Nowhere is this clearer than with speech data. With audio-taping and subsequent videotaping technology, real-time records (albeit records still made from particular points of view, not least depending on the location of the recording device[4]) could be made (and, in our digital era, made very 'cheaply'). A conventional device for rendering speech into writing is the transcript (of a record of an event). And in different discourse analytic traditions, most particularly that of Conversation Analysis (CA), a great variance in sonic detail is or is not to be included. Gail Jefferson, a colleague of sociologist Harvey Sacks, both founding figures of CA, has produced what she has termed the 'gold standard' of transcription. In her 2004 chapter, 'Glossary of transcript symbols with an introduction', Jefferson writes:

> Although I'd probably rather transcribe than any [*sic*] do any other part of the work (analysing, theorizing, lecturing, teaching, etc.), the one thing I'd rather *not* do is talk about transcribing. It's not a topic. You might as well talk about typewriting. Transcribing is just something one does to prepare materials for analysis, theorizing, etc. Do the best you can, but what is there to talk about? (p. 13; *emphasis in original*)

[4]Including the possibility of it being situated behind an ear of the teacher, thereby mirroring the teacher's eye-line. Without intending this to be an instance of product placement, see *Looxcie*.

Of course, rest assured, being an academic, she does nonetheless find a number of things to talk (write) about! But I am struck that there is no mention of calibrating the degree or extent of transcriptive fidelity of the audio recording to one's research intents and interests, even bearing in mind that one often does not fully know the nature or aspects of a research phenomenon of interest before beginning one's analysis.

In her chapter in this very book,[5] Judit N. Moschkovich concurs on this point and goes further:

> Transcription and transcript quality are theory laden (Ochs, 1979; Poland, 2002). Researchers make many decisions about transcripts that are based on their theoretical framework and on the particular research questions for a study. Decisions regarding what to include in transcripts and which transcript conventions to use are informed by theory. Whether a transcript will include gestures, emotions, inscriptions, body posture, and description of the scene (Hall, 2000; McDermott, Gospodinoff, & Aron, 1978; Poland, 2002) will depend on whether these aspects of activity are relevant or not to the particular research questions. Similarly, selecting transcript conventions and deciding whether overlapping utterances, intonation, and pauses are included or not in a transcript depends on whether these aspects are relevant to the research questions and analysis that will be carried out with the transcript and video. (Moschkovich, 2018, p. 45)

Historically, early depictions of mathematics classrooms took the form of a short narrative account of a lesson or, possibly starting around 1960 (or so!), occasional brief transcripts of teacher–student or student–student exchanges began appearing both in professional and in research journals. Questions of the veracity or fidelity of such transcripts did not explicitly feature initially, but the form of such transcripts was much influenced by the antecedent genre of play script: identified speaker turns, non-overlapping turns, occasional para-lingual or prosodic indicators, conventionalised spelling, stage directions, etc. (For more on the notion of antecedent genre, see Jamieson, 1975; for much more on recent mathematics education research employing forms of scripting, see Zazkis & Herbst, 2018.)

Staats (2008, 2018) has helpfully provided access for mathematics education to a different transcription model (one from linguistic anthropology, based on poetry rather than prose), that among other things provides a way of depicting (hence highlighting) structuring elements of repetition, highly pertinent in regard to conversations about mathematics. She writes:

> While linguistic meaning is often considered a property of words, significant mathematical ideas – arguments, inference, and relationships – can also be expressed through discourse structure. At times, the form of a student's statement can convey meaning as much as the isolated definitions of the words themselves. (2008, p. 26)

Much of this structure is marked by prosodic elements (such as rhythmic emphasis and dynamic speech variation) which may not be conventionally transcribed. The technique Staats introduces to mathematics education brings these

[5] Ah, deixis—'this very book' refers to (points at) the book I presume you, dear reader, are currently reading, not the book Jefferson's chapter is in.

elements to the fore (and allows us to see once more that *prose* transcription is a presumption). This interesting transition (or at least enlarging of transcription options), still in its infancy with respect to mathematics education (somewhat ironically, given how structured mathematical speech is), reflects a shift of what counts as significant (something I mentioned at the outset of this chapter). Who would have thought that speech rhythm and repetition (see also Tannen, 2007) could be important?[6]

4 The ICME-13 TSG 31 Panel Talk Itself (July 26th, 2016)

For my talk, I had thought I would identify a few touchstone linguistic moments from the past sixty years, both from inside and outside mathematics education. Possibilities included: John Sinclair and Malcolm Coulthard's work on classroom language from 1975, including their identification of the Initiation—Response—Feedback sequence and how it differs from Mehan's (1979) Initiation—Response—Evaluation; Paul Grice on conversational implicature, also from 1975, not to mention exploring the regular violation of his conversational maxims by teachers in mathematics classrooms; Pearla Nesher's work on the semantics of word problems from the 1970s; Julie Austin and Geoffrey Howson's seminal literature review of language and mathematics education from 1979 and, prior to that, Lewis Aiken's review from 1972 (though his 'language factors' are not the same as 'language data'); Michael Halliday's talk from 1975, introducing the notion of the mathematics register; Michael Stubbs PME 10 plenary lecture on logic and language in 1986, as well as his subsequent work on corpus linguistics (1996, 2001); Zwicky's (2003) magnificent work on metaphor; Beth Herbel-Eisenmann and David Wagner's accounts of lexical bundles in mathematics classroom talk from 2010; Andreas Ryve's meta-analytic work, from 2011, on the full extent of the decidedly smudgy use of the term 'discourse' in a wide range of mathematics education research texts (108 of them, in fact).

In passing, I feel the same about 'text' as I do about 'discourse', namely its problematic use at times to refer both to the spoken and to the written (which are sufficiently different, I feel, not to support an overarching term). And so much of striking contemporary interest is to be found in Barwell et al. (2016). And then I planned to use these pieces as potential dipsticks to try to identify changes in how language data have been approached, perceived and manipulated in published research in mathematics education (and beyond).

However, events took another course, although not as in 1992 (see footnote 2) when I was actually unable to attend the sixth ICME conference. I was in Hamburg in July 2016 and turned up on the right day and at the right time for the opening

[6]This connects more widely to issues of metaphor (as potentially central to mathematics and not just poetry)—see Zwicky (2003, 2010) and Pimm (1987, 2010). After all, it was Goethe who proclaimed, "Mathematics is pure poetry".

panel in Topic Study Group 31 (entitled "language and communication in mathematics education"). Unfortunately, I had followed the conference suggestions not to bring my own computer and had somehow managed to copy onto a memory stick only an alias of my detailed *Powerpoint* presentation (in other words, just an echo of the file was on the stick, which basically contained nothing at all). So, now (CT), I have found myself here faced with a decision: do I now write a paper that would be a pseudo-record of my intended talk that had not taken place (wow, M-tense issues all over that sentence!) or do I produce an (unpolished) written version of what I actually (well, more or less) said? I have decided to go with the latter.

However, I have also decided to do so in the form of aphoristic notes. Partly this is due to a fondness for aphoristic texts (see, for instance, Jean Baudrillard's five volumes of aphorisms, compiled at regular intervals over the space of twenty-five years, each entitled *Cool Memories*, or James Richardson's *Vectors*), and partly for other reasons (e.g. memory and its erratic ageing). Here are two aphorisms, one from each author mentioned above. (For more on aphorisms and their strengths, see Pimm, 2017.)

> Greater than the temptations of beauty are those of method.
>
> (Richardson, 2001, p. 26)
>
> All terms with negative prefixes are already stereotyped language.[7]
>
> (Baudrillard, 1995/2007, p. 88)

The panel talk slot was fifteen minutes long and seems a long time ago. Many years ago, one of my university mathematics teachers, David Fowler,[8] offered me a conference presentation meta-thought: "to say one thing that is worth saying takes two-and-a-half minutes". So, having said this aloud in Hamburg, it left me with just five more things to say.

One concerned the profound involvement of language in and with the shaping of the nature of mathematics itself, not just mathematics classroom language (see, for instance, Morgan & Burton, 2000). But there is also a curiosity in that there has been far less attention paid to the geography of mathematics as opposed to its history. For one instance of the former, concerned with the dynamic writing of fractions, see Bartolini Bussi, Baccaglini-Frank, and Ramploud (2014) and Pimm (2014a). For an extensive account of the staggering geographic and historical diversity in relation to number system notation, see Chrisomalis (2010). And Reviel Netz's (1999) important work on what he terms the 'archaeology' of the mathematical diagram highlights the effects of the devaluing of the mathematical diagram (as opposed to the 'text', as if diagrams too were seen as not 'written') of mathematical proofs. This is all perhaps the result of a combination of the illusion of and the imposition of a sense of mathematical 'universality'.

[7]What particularly comes to mind for me from this aphorism are such terms with negative definitions, like *irrational* or *non-recurring* or *discontinuous*.

[8]See Pimm (2004).

A second point involved how mathematics also leaves (potentially unaware) traces in us: gestures, rhythms, body counting, and so on—traces that may not be as contemporary as the words that accompany them. It is possible for one's hands to be in one century and one's vocal cords in another. (For an example of this, though not explicitly framed as such, see Núñez, 2004/2006.) And certain technology (e.g. touchpad dynamic geometry environments) might provide ways of returning one's hands to a previous century, such as by heading back to the seventeenth-century notion of variable in regard to creating diagrams/graphs (as opposed to graphs being an already-existing static set of points satisfying a relation). For more on the connection between mobility, gestures and diagrams in mathematics, see Châtelet (2000) and Ng (2015).

This is closely connected to my third point which was (of course) about the mathematics register. How it can mess with the mainstream grammar of the language: the tension between nominalisation (everything in formal, written mathematics ends up as a lifeless, timeless noun) and verbification (the return of time and human action—see Lunney Borden, 2011). But professional mathematicians and scientists talking informally to each other are much more informal and casual in their language use (see Barwell, 2013; Ochs, 1979). Mathematics is both static and dynamic (and dynamic geometry software is supporting the return of 'time and motion' studies[9] as part of mathematics) and this connects to a core distinction

[9]In the early part of the twentieth century, there was considerable interest in the notion of 'arrested motion' in sculpture, endeavouring to capture the dynamic in the static. In a *Paris Review* interview, writer William Faulkner asserted:

> The aim of every artist is to arrest motion, which is life, by artificial means and hold it fixed so that 100 years later when a stranger looks at it, it moves again since it is life. […] This is the artist's way of scribbling […] oblivion through which he must someday pass. (1956, pp. 49–50)

This contrasts interestingly with historian of science Catherine Chevalley's comments about her father Claude, a core Bourbaki mathematician:

> For him [Claude Chevalley], mathematical rigour consisted of producing a new object which could then become immutable. If you look at the way my father worked, it seems that it was this which counted more than anything, this production of an object which, subsequently, became inert, in short dead. It could no longer be altered or transformed. This was, however, without a single negative connotation. Yet it should probably be said that my father was probably the only member of Bourbaki who saw mathematics as a means of putting objects to death for aesthetic reasons. (in Chouchan, 1995, pp. 37–38; *my translation*)

There is also a significant sense in which language—not least mathematical language (especially the written)—provides a means for arresting motion:

> a point is an instance,
> arrested motion – the geometric,
> its unsigned art.

between definition by genesis and definition by property (see Molland, 1976; for a piece using this distinction within mathematics education, see, for instance, Chorney, 2017). Syntax matters. And it shapes our perception of mathematics, something which becomes indistinguishable from mathematics itself.

My fourth point concerned language data, which I have already addressed to a certain extent at the outset of this chapter.

The last one (which I did not get to say aloud) I had labelled *être et avoir*. This is, among other things, the title of a French documentary film about a year in a small rural elementary school with a very mixed-grade class (aged from four to twelve), but is also a binary classification (based on auxiliary verbs) of verbs in French, with respect to how the perfect tense is formed. In English, both the verbs ‘to be’ and ‘to have’ show up in mathematics a lot and convey different messages about the mathematical object or result under consideration. How does the copula ‘to be’ (which sounds so active) actually deep-freeze the mathematical world?

5 In Conclusion

A few years ago, I began a ZDM commentary piece as follows.

> Part of the opening paragraph of the description for a doctoral course I am teaching runs as follows:
>
> Almost every piece of mathematics education research is based on language data to some greater or lesser extent, where ‘language’ needs to be more or less broadly interpreted. Whether these data arise from oral interviews, transcripts of classroom video recordings, textbooks (ancient or modern), student written responses to tasks, mathematicians’ writing or teacher study group recordings or …, you need to be able to work with and analyse language data at length and at depth. I am particularly interested in questions of *method* and the *manner* of data collection/generation/creation, their examination and analysis.
>
> As you can see, I am interested in the varied uses the field makes of language as data, and this is how I propose to open my remarks here. But before that, I start with a question. When did research articles in mathematics education start including elements of language as data? Some fragment of classroom language perhaps or possibly an extract of clinical interview data or maybe student written responses to test questions or even a textbook page in a curriculum study? I ask because it clearly does have an origin as a practice (as well as a possibly earlier genesis in professional journals) and I am curious about not just when but why. (Pimm, 2014b, p. 967)

Four years on, I am pretty much still where I was—interested, but not much further forward (and, of course, preparing to teach that same course in January 2018). An enormous amount of data in mathematics education actually is language data. (Work on gestures may be one contemporary exception, though by incorrectly labelling this phenomenon as ‘para-linguistic’ a certain monster-adjustment of my primary claim may be made. Gestures are certainly a key element of mathematical communication.) Consequently, how language data is conceived and handled is of the utmost significance to our field.

Acknowledgements I am grateful to David Wagner and Judit N. Moschkovich for comments on earlier drafts, as well as Nathalie Sinclair, my ideal reader.

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Subject-Specific Academic Language Versus Mathematical Discourse

Marcus Schütte

Abstract The significance of language for the learning of mathematics has long been thematised in mathematics education research. Since Austin and Howson provided the first overview of the state of research in 1979, the field has become more differentiated. The present article will discuss one area of research emerging from this differentiation—multilingual contexts. This example shows how mathematics and language as a research field has developed from dichotomous approaches towards the idea that the language of mathematics is characterised differently in different cultural and group contexts, thus emphasising discursive aspects. This trend gives rise to the question of how the individual resources of participants can be acknowledged and exploited in groups with different abilities, while simultaneously providing the participants with the necessary linguistic support to participate in the linguistic discourse of that group.

1 The Development of a Linguistic Perspective on the Learning of Mathematics

The significance of language for the learning of mathematics has long been a topic of international discussion in the area of mathematics education. Austin and Howson (1979) were among the first to provide a summary of research in the field; since then, mathematics education as a research field has evolved, and diverse papers have been published on the topic of mathematics and language, bringing various aspects into focus. In any discussion of mathematics and language, the extraordinary work of David Pimm cannot be ignored. Pimm's pioneering work has been raising awareness of mathematics and language as a research topic over a 30-year period. His book *Speaking Mathematically* is based on the importance of language in general and in mathematics education in particular. In it, he lays the foundations for later research efforts in this field. According to Pimm (1987), the

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© Springer International Publishing AG, part of Springer Nature 2018
J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_3

learning of mathematics is linked to the learning of language. Pimm sees mathematics as similar to a foreign language which certain people cannot speak, albeit one clearly differentiated from natural languages like German or English. Pimm states that mathematics is not a solely written language but also a spoken one, which has to be used extensively within the mathematics classroom. In this perspective, the teacher is similar to a "native speaker" in a natural language (cf. Morgan, Craig, Schütte, & Wagner, 2014). In the German-language literature, the first to engage comprehensively with these ideas were Maier and Schweiger (1999), in their book *Mathematik und Sprache*. Above all, they postulate that the learning of mathematics is in large part a question of being introduced to a subject-specific mathematical language. According to Morgan (2014, p. 389), such a language is characterised by the following features:

- "special vocabulary used to name mathematical objects and processes"
- "the development of dense groups of words such as lowest common denominator"
- "the transformation of processes into objects".

According to Morgan, specialised domains of activity have their own specialised vocabularies and ways of speaking and writing. Pimm (1987) makes reference to the existence of a mathematical register in the English language (or any other language). Registers are specialized uses and meanings of a specific language for mathematical purposes (e.g., specialized meanings and purposes for vocabulary (words, phrases or expressions) as well as grammatical structures) that can be chosen by an individual to fit a situation or a context. Thus, a register is clearly different from a dialect, which is usually limited to a specific geographical region. Using or developing a register, according to Pimm, is not only a question of using technical terms; it is also about using certain phrases and characteristic modes of arguing (p. 76). In this context, Pimm draws on Halliday's (1975, p. 65) definition of a register as "a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings" (Pimm, 1987, p. 75). Halliday says about the "mathematical register" that one can refer to it "in the sense of the meaning that belongs to the language of mathematics" (Pimm, 1987, p. 76). The mathematical register, like any other linguistic register, consists of words, phrases, and expressions borrowed from the English language (or any other language) and of terms that are solely created to describe something that only exists in mathematical contexts and has no meaning or different meaning outside of those. Pimm elaborates the creation of a mathematical register. For him, one important possibility is using the role of metaphors as a tool to create meaning. Pimm speaks of extra-mathematical metaphors and structural metaphors, seen as the two "main sources of metaphor" (Pimm, 1987, p. 93) to construe the mathematical register. The former are used to "explain or interpret mathematical ideas and processes in terms of real-world events […], e.g. a graph is a picture", while the latter involve "a metaphoric extension of ideas from within mathematics itself" (Pimm, 1987, p. 95). In accordance with Hymes (1972), Pimm attaches a particular importance to “communicative competence” (p. 4) in mastering a language or language style. This is the ability to use language in social situations, i.e., in context. Like Halliday (1989), Pimm sees the learning of language as a process that depends on the language not being isolated from the context; indeed, it is always in context that the language must be invested with meaning. Only through learning language in context do learners become able to apply different linguistic styles appropriately in the respective situation: “Communicative competence, then, involves knowing how to use and comprehend styles of language appropriate to particular social circumstances” (Pimm, 1987, p. 4).

Although the dependence of mathematical language on respective contexts was thus postulated at an early stage, the contrasting notion of the universality of this language persisted for a long time—and unfortunately can still be seen in many studies today. However, comparison studies carried out in various cultures consistently concluded that the notion of mathematics as a universal language, a language that could be learnt regardless of cultural or native-language influence, was far less frequently appropriate than often being suggested. Recognition of the culture- and thus also language-dependent nature of mathematics, i.e., that native speakers of mathematics in fact speak this “language” differently in different cultures or groups, began to gain traction in the context of the increasing influence of the “ethnomathematical” stance (cf. D’Ambrosio, 1985). This approach emphasises that mathematics is sensitive to cultural idiosyncrasies, including those related to language (cf. Morgan, 2014), which has been confirmed in recent studies. This development occurred on a background of what Lerman (2000) calls the “social turn” of research on teaching and learning in mathematics education. The social turn, which appeared around the end of the 1980s, describes a development within mathematics education research where “the social origins of knowledge and consciousness” (p. 8) have increasingly been taken into consideration. This is not meant to imply that social factors have previously been ignored by other theories like Piaget’s theory of learning or radical constructivism, as they saw social interactions as stimuli for meaning-making within an individual. However, social activity was now increasingly seen as the source producing meaning, thinking, and reasoning. Lerman suggests three primary disciplines which contributed to the development of the social turn: anthropology, sociology, and cultural psychology. Essentially, the social turn no longer meant studying a person and his or her meaning-making separately, but taking into account the person’s actions within a social practice. This trend turned mathematics education into a mature field of study in which serious efforts are now being made towards the theorisation and problematisation of components, concepts, and methods, including language. Increased sensitivity to the role of the social environment within which mathematics education takes place has inevitably meant greater attention to language and all forms of communication within mathematical learning environments (cf. Morgan et al., 2014). Apart from the studies by D’Ambrosio, those by Bishop (1988), Cobb (1989), Lave and Wenger (1991) and Wertsch (1981) should be mentioned here.

In German-language research, the social turn was brought into clearer focus within the discussion through diverse studies using interactionist approaches of interpretive classroom research, for example those by Bauersfeld, Krummheuer and Voigt (see, among others, Bauersfeld, Krummheuer, & Voigt, 1988; Krummheuer, 1995). These studies explicitly renounce the previously dominant view that learning was merely an internal psychological phenomenon. The social turn and the inclusion of interactionist aspects of learning and teaching meant a shift(ing) of focus from the structure of objects to the structures of learning processes, and from the individual learner to the social interactions between learners (cf. Bauersfeld, 2000).

The transformed understanding of learning led to the development of theories that see meaning, thinking and reasoning as products of social activity. These are evidenced, for example, in the above-mentioned interactionist approaches of interpretive classroom research. Thus, Krummheuer (1992, 2011) argues that learners are involved in "collective argumentations" in the learning of mathematics in primary school (Krummheuer, 1992, p. 143) and it is through their increasingly autonomous participation that they learn mathematics (cf. Krummheuer & Brandt, 2001). Similar ideas are reflected in recent work by Sfard (2008). Underlining the significance of language for subject-specific learning, she writes that thinking is a form of communication and that learning mathematics means modifying and extending one's discourse. So, following Krummheuer (2011), mathematics learning can be described as the "progress" of participation emerging from the coordination of interpretations in collective forms of argumentation. Sfard (2008, p. 92) suggests replacing the notion of "learning-as-acquisition" with that of "learning-as-participation" (p. 92). Lave and Wenger (1991, p. 35) conclude that the start of this kind of acquisition process can be understood as a "legitimate peripheral participation" (in German, "legitime periphere Partizipation"). A learning process can then be described on the interactional level as a path from legitimate peripheral participation towards being a "full participant" (p. 37). Based on the fundamental assumption of these approaches, i.e., that meaning is negotiated in interactions between several individuals and that social interaction is thus understood as constitutive of learning processes, language can no longer be only understood as the medium in which meaning is constructed; rather, speaking about mathematics in collective argumentations is in itself to be seen as the "doing" of mathematics and the development of meaning. Thus, language acquires a central significance, if not *the* central significance in the building of mathematical knowledge and the development of mathematical thought.

Following Morgan et al. (2014), we could summarise that research on language in mathematics education has moved from a primary focus on mathematical words or symbols towards a more comprehensive engagement with a range of other means of communication. With the new emphasis on the social environment, researchers have begun to concentrate on face-to-face communication in classrooms, that is, on speaking more than on written texts. This has led to the recognition that the spoken discourse of the mathematics classroom also has unique characteristics. In their recent overview of what is now a diverse area of research, Morgan et al. (2014,

p. 864) name four subfields under which the current research efforts can broadly be categorised:

- analysis of the development of students' mathematical knowledge
- understanding the shaping of mathematical activity
- understanding processes of teaching and learning in relation to other social interactions
- multilingual contexts.

Concerning the development of students' mathematical knowledge, past research analysed the language of students drawing a direct link to their mathematical knowledge, viewing language as an unambiguous medium for transmitting ideas. However, this conception of language has been challenged on the basis of new insights about language and communication, based for example on the semiotics of Peirce, Wittgenstein's idea of language games, and post-structuralist theories, which object to the existence of a stable relationship between a word and its referent. These theories have been integrated and developed further within mathematics education research, thus adding to notions of how mathematical learning can be understood.

One example of the inclusion of semiotics into mathematics education research is Steinbring's (2006) theoretical concept of the epistemological triangle, which can be used to describe the way mathematical knowledge is developed. This is done by focusing on the relationship between representations of mathematical concepts (e.g., symbols, words, etc.) and the concepts themselves, as well as on how students' previous knowledge and experience ("reference context") mediates these relationships. The second subfield, which focuses on the shaping of mathematical activity, follows a different approach when looking at how mathematical knowledge is developed. Here Vygotsky's perception of verbal language and other semiotic systems as tools affecting human activities is used as a basis. An example of this is the communicational theory of Sfard (2008), which does not differentiate between communicating in mathematical forms and doing mathematics/thinking mathematically. The way learners engage in mathematical activities can, therefore, be described by characterising the nature of mathematical language in detail. Using tools like Systemic Functional Linguistics or other tools which enable tracing the development of ideas or meanings within language practices help understand how students obtain their conceptions of mathematics which are carried into adulthood (e.g., Chapman, 2003). These kinds of analyses can, for example, help identify how people may be empowered with mathematics through attention to language (cf. Wagner, 2007) or through particular language practices. This stresses the significance of language aspects for learning mathematics not only for young learners in school but also for adults later in life. The third subfield is concerned with understanding processes of teaching and learning in social interactions. Here, studies in mathematics education are connected to the general notion which sees learning as a social activity and meaning-making no longer as the activity of an individual but of an individual within a social environment. By using tools for analysing classroom interaction which often originate outside of mathematics education—for example, in ethnomethodology or linguistics—patterns within social interactions can be identified. These patterns—for example the funnelling pattern identified by Bauersfeld (1988)—have proven to be useful when working with teachers. In Germany, other recent studies in this field of interactionist approaches of interpretive classroom research in mathematics education include, for example, those by Brandt (2013), Fetzer and Tiedemann (2015), Krummheuer (2011, 2012), Meyer and Prediger (2011), Schütte and Krummheuer (2013) and Schütte (2009, 2014). Specific mathematical issues which emerge when analysing interaction within mathematics classrooms include socio-mathematical norms (Yackel & Cobb, 1996) and forms of interaction which are specific to mathematics, like argumentation (Krummheuer, 1998; Planas & Morera, 2011) or group problem solving.

Primarily, however, I will focus on the last of these subfields. Up-to-date international comparison studies are underway in this area concentrating on lower academic achievement and reduced educational opportunities among children with a migration background in many European countries; the present migration flows into Europe have also brought it into the focus of the current social and scientific discussion. The following will examine in more detail the research efforts on multilingual contexts and the developments in the field, in order to determine the tasks such research needs to address.

2 Multilingual Contexts—From Deficit to Resource

In their overview of the research field of mathematics and language, Austin and Howson (1979) also address multilingual contexts. They mention various studies from the end of the 1960s and the start of the 1970s, which present a rather fragmented picture. Alongside studies that address the topic in the context of cultural and linguistic diversity conditioned by migration, they also find studies that focus on learning among minorities in school systems that are characterised by majority groups. Furthermore, they dwell in particular on diverse studies that concentrate on learners who are taught in developing countries not in their native language but in the respective administrative languages, for example English. Their results appear just as diverse as the data they are based on. They point to positive effects of a bilingual education in developing countries in particular, though these results are partially contradicted by data gathered in other countries. Overall, a rather positive picture of learning under bilingual conditions is presented. Some studies show that bilingual children may well be in advantageous positions when compared to monolingual children (e.g., Gallop & Kirkman, 1972). The following decades, however, saw a change in this positive perspective on learning under conditions of cultural and linguistic diversity.

Around the end of the 1970s, an approach developed by Cummins in the context of English second-language acquisition began to gain influence in the teaching and learning research community in relation to learning under bi- or multilingual conditions. Cummins' (1979, 2000) differentiation between "basic interpersonal communicative skills" (BICS) and "cognitive academic language proficiency" (CALP) had a fundamental influence on the consequent discussion in mathematics education. In Cummins' work, "BICS" refers to fundamental communication and language abilities in everyday communication and interactions. These are situation-dependent and are understood in an informal context. Meanwhile, "CALP" represents special school-related cognitive language knowledge and abilities, which are relevant for example in subject-specific discourse in the classroom. An essential aspect of Cummins' concept is the idea that children can quickly gain abilities in their second language which they can use in everyday situations, but need significantly longer to achieve the competences in the academic language of the classroom that are required for academic success. With this conceptualisation, the positive perspective on cultural and linguistic diversity appears to fall away, with the focus placed instead on the fact that some children learn academically relevant linguistic competences more slowly than others, or not at all. This suggests a closeness to Halliday's (1975) register-based approach, mentioned above, which also influenced Pimm (1987). Pimm translates this approach more for mathematics in general, but in using the metaphor of the native speaker of mathematics he also distinguishes between natives and non-natives in terms of deficits. However, he emphasises that more than speaking like a mathematician, the point is to learn "to mean like a mathematician" (p. 207); this suggests a closeness to the above-mentioned approaches which consider the negotiation of meaning in interactions to be significant for the linguistic learning of mathematics. Cummins' perspective, meanwhile, clearly focuses on a kind of "target register" which all children must learn to use in order to successfully participate in mathematics teaching. In the following years, this approach has influenced the emergence in the research field of a perspective focusing on children's linguistic deficits above all, looking at the differences between children's abilities and the demands of the target register. In Germany, a discussion developed around children's abilities in an academic language (cf. the concept of "Bildungssprache", Gogolin, 2006) in schools, seen as abilities which children need to master in order to be academically successful, resonating with Cummins' concept. A definitive characteristic of this academic language is its conceptual written form, which means it shows high information density and context-independence, and fundamentally does not correlate with the features of the everyday oral communication engaged in by many pupils (cf. Gogolin, 2006). Various authors point out that this independence from the present situation—often described in terms of the ability to decontextualise—represents a fundamental characteristic of the school discourse based on academic language (cf. Bernstein, 1996; Cloran, 1999; Gellert, 2011).

Current approaches in international research on mathematics education criticise this narrow view of mathematics learning from a linguistic perspective. Aukerman (2007), for example, argues that the ability to decontextualise language, i.e., to separate language from its context, as emphasised by Cummins, needs to be re-thought. The CALPS concept is focused on leading children step by step from context-oriented language towards a language that is more context-independent.

Thus, context is seen as a transient step towards an academic learning. But for Aukerman, it does not make sense to talk about a decontextualisation of language. A language will always remain incomprehensible to a child if he or she cannot find a meaningful context; this is always determined by what the child already knows and trusts. Therefore, she suggests the alternative concept of "re-contextualisation." According to this, children acquire language that they need in order to carry out a range of academic and non-academic tasks. To do this, they must use the support of linguistic competencies they already have (including the non-academic competencies) and transform these competencies in new contexts. For Aukerman, the task of the teacher is therefore primarily to support children in their "recontextualisations," acknowledging the children's ways of thinking, and using this starting point to work together with the children to make new academic material meaningful and relevant to them.

In the German-language literature, Prediger (2002) uses the approaches of Bauersfeld (1988) and Maier and Voigt (1991) to consider communication processes in mathematics learning as intercultural communication. Her basic position is that all mathematical communication with learners is intercultural communication. The teacher functions as a representative of the mathematical culture while the learners apply their everyday culture—this suggests a closeness to Pimm's (1987) notion of native speakers of mathematics. Prediger distances herself from a deficit-oriented perspective, emphasising the importance of acknowledging differences among learners without imposing values. Further similarities with Pimm can be found in the way she sees the central barriers to communication not only in individual problems or problems emerging through interactions, but above all in problems rooted in the subject-specific culture of mathematics (Prediger, 2002, p. 1).

Another link to interactionist approaches of interpretive classroom research can be observed here, which Prediger acknowledges. According to Krummheuer (1992), participants in classroom interaction interpret activities in extremely diverse ways, based on their different abilities and backgrounds. Following this interactionist perspective of mathematical learning, an individual context-dependent meaning is negotiated for the content of the respective situation in the interplay between connected oral contributions and accompanying actions. This is referred to as the "situation definition" (Krummheuer, 1992, p. 22). Situation definitions are constantly adapted and transformed with other participants in the interaction in "negotiations of meaning," such that the process of the production of interpretations is never concluded. The production of "simple" situation definitions does not necessarily lead to new learning, however.

Standardised and routinised individual situation definitions are termed by Krummheuer (1992, p. 24) as "framings", drawing on Goffman's concept of "frame" (1974, p. 19). Pupils' framings are often not in agreement with those of other pupils or the teacher (cf. Krummheuer, 1992). In this perspective, teaching can be understood in terms of the "juncture of framings from two different interactional practices, which itself becomes practice" (ibid. p. 64, translated by the author). Differences between framings can be explained in Pimm's (1987) terms as differences between interactional practices of native and non-native speakers, and in

Prediger's (2002) as differences between those representing the subject-specific culture of mathematics and those representing the everyday culture. It is only the fundamental transformation or construction of framings that represents a learning process here, not the transformation of situation definitions (cf. also Schütte, 2009). According to Krummheuer, differences in framings between the participants, which obstruct the production of collective argumentations but at the same time represent the motor of learning, need to be increasingly coordinated by the individual with advanced skills in the interaction (usually the teacher).

With her "situated-sociocultural perspective", Moschkovich (2002) also emphasises the significance of discourse in the learning of mathematics. The perspective completes a switch from a consideration of obstacles and deficits of learners to one of resources and competences of a diverse pupil population (cf. Planas & Civil, 2013, as well as the contrary results of Meyer & Prediger, 2011). In this perspective, the learning of mathematics always takes place in a public social and cultural context, and represents a discursive activity. However, there is not one correct mathematical discourse that needs to be achieved, contrary to what approaches based on the concept of register often suggest. Learners participate in mathematical discourses in different communities, using diverse resources from different registers in order to communicate successfully mathematically (Moschkovich, 2002). In contrast to register-based approaches, the concept of mathematical discourse makes clear that interactional or non-language aspects must take a central role in the understanding of the learning of mathematics.

3 Summary

In mathematical learning situations, it is perhaps not surprising that comprehension problems can emerge specifically among children with relatively unschooled or multilingual backgrounds as a result of migration, in classrooms where teaching is not accomplished in their native tongue and is directed by a native speaker of mathematics (Pimm, 1987), i.e., a representative of the mathematical culture (Prediger, 2002). In the context of the current discussion on children's linguistic competences in school, this could be explained on the basis of children's linguistic deficits. Many authors see a solution in training children's linguistic competences in order to redress the presumed linguistic deficits in academic language (Cummins, 1979, 2000; Gogolin, 2006).

Potential misunderstandings in learning situations can also be explained by different interpretations of situations based on differing framings among participants (cf., Aukerman, 2007; Krummheuer, 1992; Moschkovich, 2002; Schütte, 2014), underlining the interactive aspect of doing mathematics. Significantly, however, according to Schütte (2014), framings of situations that differ from the framing of the teacher (or adult advanced in the interaction) can be reconstructed not only in children with presumed linguistic deficits, but also in children with monolingual and relatively schooled backgrounds. This enables us to hypothesise that the framings of children with clearly differing linguistic competencies can nevertheless be very close to each other, and that the framings of children with less linguistic deficits could develop just as large a difference to the framing of the teacher. This would resonate with Pimm (1987) and Prediger (2002) in suggesting that a perspective focusing exclusively on children's possible linguistic deficits comes up short. It is certainly desirable for all participating children to be introduced to formal and subject-specific mathematical language aspects, and for the teacher to act explicitly as a linguistic role model. But even when children have a linguistic role model, they need a teacher who engages with their interpretations and tries to modulate the basic framings of all participants to enable "learning despite differences".

Due to the increasing diversity of pupil populations, situations of multiple different interpretations in negotiations of meaning in the classroom will become more and more prevalent. In spite of this diversity, children's basic framings, emerging from an everyday life that is at least partially shared, might have more in common with each other than with the subject-specific framing of the teacher based on his or her training in mathematical education—and they need to be appropriately coordinated. The goal should be firstly to accustom teachers to such diversity of interpretations of taught content, and, building on this, to develop their interpretive competency to recognise differences based on different framings, thematise them in the learning process, and thus produce the possibility of modulation. This is not to argue that teachers should not provide children with linguistic support. However, it seems that one future task of mathematics teaching will entail using children's linguistic resources positively, for example allowing them to switch into their first language during group work, as well as providing them with opportunities to build linguistic competences in the principal teaching language.

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Recommendations for Research on Language and Learning Mathematics

Judit N. Moschkovich

Abstract This paper describes recommendations for research on language and learning mathematics. I review several issues central to conducting research on this topic and make four recommendations: using interdisciplinary approaches, defining central constructs, building on existing methodologies, and recognizing central distinctions while avoiding dichotomies. I make four recommendations to address these issues.

1 Introduction

Researchers in mathematics education who address issues of language have used work from fields outside of mathematics education to inform research on the relationship between language and mathematics learning. Work outside of mathematics education has contributed theoretical frameworks for studying discourse in general, methodologies (e.g., Gee, 1996), concepts such as registers (Halliday, 1978) and Discourses (Gee, 1996), and empirical work on classroom discourse (e.g., Cazden, 1988; Mehan, 1979). While concepts and theories from other disciplines provide essential resources, borrowing concepts also presents challenges. There is danger in borrowing concepts and leaving behind the intellectual tradition that gives a concept its meaning. Notions such as language, register, or discourse are complex: these terms have contested meanings, long histories, and are the topics of debates in other disciplines such as anthropology and linguistics. When using these constructs in mathematics education research we need to respect the traditions surrounding those constructs and apply them carefully to mathematical settings. Similarly, methodologies from fields outside of mathematics education provide a basis for work that examines language and mathematics learning/teaching. These

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© Springer International Publishing AG, part of Springer Nature 2018

J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_4

methodologies may need to be re-worked as they are applied to mathematical settings, for example so that they can serve to document classroom discourse about mathematical ideas and by focusing not only on language but also on mathematical activity.

2 Recommendation #1: Draw on Interdisciplinary Approaches and Methodologies

Research on language and mathematics education must be grounded not only in current theoretical perspectives of mathematical thinking, learning, and teaching but also in current views of language, classroom discourse, bilingualism, and second language acquisition. Research needs to consider interdisciplinary approaches, use frameworks for recognizing the mathematical reasoning learners construct, and consider multiple methods for data collection and analysis.

Since mathematical activity is multi-modal and multi-semiotic (O'Halloran, 2000), and mathematical understanding involves multiple modalities and artifacts—including oral and written language, gestures, the body, inscriptions, and so on—the study of language and mathematics requires interdisciplinary approaches. Research on this topic has used several approaches—such as situated cognition, anthropological, cultural historical activity theory, systemic functional linguistics, applied linguistics, Goffman's notion of frames, discursive psychology, and embodied knowing—and there are many more. It is important to draw on relevant studies, even when these studies are from different content areas. For example, studies focused on science classrooms and discourse (e.g., Lemke, 1990; Warren, Ogonowsky, & Pothier, 2005) may be relevant to research in mathematics classrooms.

3 Recommendation #2: Define Central Theoretical Constructs and Connect Those to a Theoretical Framework

Overall, research studies need to be clear and explicit as to how the term language is defined and used. Uses of the term language refer to a spectrum of phenomena, ranging from the language used in classrooms, to the language used in the home and community, or the language used by mathematicians, or the language in textbooks, or the language in test items. It is crucial that mathematics education researchers clarify how this term is defined, what set of phenomena it refers to, and which aspects of these phenomena are the focus of a study.

Research studies need to draw on a rich and complex understanding of what "language" is, utilize interdisciplinary theoretical approaches and methods, and consider language issues more broadly as they function in multiple settings. As a start, research needs to: (a) recognize the complexity of language use (in classrooms or other settings) and the need to explore language in all its complexity; (b) move beyond simplified views of language as words or vocabulary; (c) embrace the multimodal and multi-semiotic nature of mathematical activity; and (d) shift from monolithic views of mathematical discourse and dichotomized views of discourse practices.

Research needs to move beyond oversimplified views of language. An emphasis on vocabulary and formal language limits the linguistic resources teachers and students can use in the classroom to learn mathematics with understanding. Work on the language of specific disciplines provides a complex view of mathematical language (e.g., Pimm, 1987) as not only specialized vocabulary or lexical aspects (new words and new meanings for familiar words) but also as extended discourse that includes syntax and organization (Crowhurst, 1994), the mathematics register (Halliday, 1978), and discourse practices (Moschkovich, 2007).

The terms "discourse" and "register" can also be used with multiple meanings. Discourse is often used to refer to classroom discussions, or to talk, or only to oral communication. We need to be clear when we use different meanings for *discourse*. For example, the label *discourse* can be used to refer only to talk. Or it can be used to mean more than talk, and include non-talk modes of participation, but also text, as well as gestures, gaze, and posture. Interpretations of "register" need to move beyond interpretations of the mathematics register as merely a set of words and phrases that are particular to mathematics. The mathematics register includes styles of meaning, modes of argument, and mathematical practices and has several dimensions such as the concepts involved, how mathematical discourse positions students, and how mathematics texts are organized.

Research needs to move from viewing language as autonomous and instead recognize language as a complex meaning-making system. To embrace the nature of mathematical activity as multimodal and multi-semiotic (O'Halloran, 2000), research needs to expand beyond talk to consider the interaction of the three semiotic systems involved in mathematical discourse—natural language, mathematics symbol systems, and visual displays. In particular, studies will need to examine how artifacts serve as mediators and how mathematical activity is embodied (Gutiérrez, Sengupta-Irving, & Dieckmann, 2010).

Research also needs to make a shift away from conceiving *mathematical discourse* or *mathematical practices* as uniform. Mathematical discourse is not a singular, monolithic, or homogeneous practice or set of practices. It is a system that includes multiple forms and ranges over a spectrum of mathematical discourse practices in different settings and contexts, such as academic, workplace, playground, street-selling, home, etc. Many more research studies are needed to better understand how mathematical practices and discourses differ depending on the setting, context, and circumstances. In particular, studies need to consider what mathematical knowledge and discourse practices learners use in different settings, what knowledge and discourse practices learners use across settings, and how to make visible the ways that learners reason mathematically across settings. Instead of asking general questions such as "Does language impact mathematical reasoning?" research needs to ask how, when, and under what circumstances are language and mathematical reasoning connected, and consider the multiple ways that language functions in different circumstances and for different aspects of mathematical reasoning. In documenting mathematical practices across settings, researchers should consider the spectrum of mathematical activity as a continuum, instead of reifying the separation between practices in out-of-school settings and the practices in school. Analyses should consider every-day and scientific discourses as interdependent, dialectical, and related rather than assume they are mutually exclusive. Rather than debating whether an utterance, lesson, or discussion is or is not mathematical discourse, studies should instead explore the multiple meanings that practices, inscriptions, and talk might have for participants or how participants use practices, inscriptions, and talk as resources to accomplish their goals.

We also need to be thoughtful when defining and using the constructs "text" and "literacy." Mathematics texts are not only writing in words or symbols (expressions and equations), they also include, but are not limited to, graphs, diagrams, proofs, justification, manipulative displays, calculator read outs, written descriptions of problem situations, as well as verbal mathematical discussions (Siebert & Hendrickson, 2010). Word problems are a particular genre that involves not only the word problem text as given, but also students' descriptions of their solution pathways, explanations for their solutions, and multimodal or graphic representations of the meaning of the problem or solution.

4 Recommendation #3: Build on Existing Methodologies to Focus on Both Language and Mathematical Activity

In order to focus on the mathematical meanings that learners construct rather than the mistakes they make or the obstacles they face, researchers need to use methodologies for recognizing the mathematical reasoning that learners are constructing in, through, and with language. There are multiple theoretical frameworks available to accomplish this, including systemic functional linguistics (e.g., O'Halloran, 2000; Schleppegrell, 2010), a communication framework for mathematics instruction (Brenner, 1994), and cultural-historical-activity-theory (Cole & Engestrom, 1993). Methodological possibilities range from ethnographic studies, to grammatical analyses, to multiple approaches to discourse analysis, to conversation analysis. Although methodologies from fields outside of mathematics education provide a basis for work that studies language and mathematics learning/teaching, these methodologies need to be re-worked as they are applied to mathematical contexts so that they serve to document classroom discourse by focusing on both language and the mathematical content and meanings in activity. One way that these methodologies may be re-worked is to focus more on mathematical activity (and how that is defined). Another way that mathematics education researchers might re-work those methodologies would be to be explicit about how their contexts and orientations differ from those of the original creators of those concepts or tools, and to describe what those differences might mean for a particular analysis. In those ways, researchers would be explicit about how the analyses we do relate to the mathematics learning in particular.

5 Recommendation #4: Recognize Central Distinctions, but Avoid Dichotomies

Some distinctions are fundamental to work on language and communication. However, there is a difference between a distinction and a dichotomy. A distinction highlights a difference or contrast between similar things. A dichotomy creates a rigid division between two things that are assumed to be in opposition, or entirely different, or always in conflict. Research should move away from dichotomies that create unproductive and oversimplified approaches to research phenomena. In this section I describe several distinctions relevant to work on language in mathematics classrooms: Conversation analysis/discourse analysis, everyday/academic language, and kinds of communication. One important dichotomy to avoid is that between qualitative and quantitative approaches. Current work in mathematics education has shown that it is advantageous to combine qualitative and quantitative approaches, either in one study or across studies. For example, Kazemi and Stipek (2001) combined the two approaches across studies; they describe quantitative analyses, then used classroom discourse selections for further micro-genetic qualitative analysis. Similarly, Zahner, Velazquez, Moschkovich, Vahey, and Lara-Meloy (2012) describe the quantitative analyses that provide the basis for their qualitative analysis of selected lessons, including a graph of how students engaged with the materials over time during a lesson.

5.1 Conversation Analysis and Discourse Analysis

The distinction between conversation analysis (CA) and discourse analysis is subtle. This distinction is relevant for mathematics education because there are pitfalls in mixing up traditions. Importing one concept from one tradition (for example, repair from CA) while using the theoretical lens or data of another (anthropology and ethnography) can create conflicting theoretical assumptions.

Conversation analysis grew out of ethnomethodology (Garfinkel, 1967). CA researchers use audio or video recordings of naturally occurring phenomena, that is non-experimental, interactions as their data. They do not use interview data, observational data or field notes, idealized or invented examples based on a researcher's own intuitions, or any other experimental methodologies. Some central concepts include turn taking, repair, and adjacency pairs. CA analyzes talk during interactions by examining only recordings (audio or video) and does not include written text or larger social or cultural phenomena.

The label "discourse analysis" has multiple meanings depending on the researcher, country, and context. Typically, data include interview data and/or ethnographic observation data or field notes. One example is the ethnography of communication (Hymes, 1964), a method of discourse analysis that draws on anthropology and ethnography. It provides a framework for analyzing communication from an ethnographic perspective, considering the context, beliefs, social and cultural practices of particular speech communities. This model provides the components to consider for describing a speech event: setting, participants, purposes, sequences, tone, modes, norms, and genres:

S *setting and scene*: where the speech event is located in time and space;
P *participants*: who takes part in the speech event, and in what role (e.g. speaker, addressee, audience, eavesdropper);
E *ends*: what the purpose of the speech event is, and what its outcome is meant to be;
A *act sequence*: what speech acts make up the speech event, and what order they are performed in;
K *key*: the tone or manner of performance (serious or joking, sincere or ironic, etc.);
I *instrumentalities*: what channel or medium of communication is used (e.g. speaking, signing, writing, drumming, whistling), and what language variety is selected from the participants' repertoire;
N *norms of interaction*: what the rules are for producing and interpreting speech acts;
G *genres*: what 'type' does a speech event belong to (e.g. interview, gossip), and what other pre-existing conventional forms of speech are drawn on or 'cited' in producing appropriate contributions to talk.

5.2 Every-day and Academic Language

The distinction between everyday/academic, formal/informal, or in-school/out-of-school seems intuitive and is sometimes credited to Vygotsky. However, this distinction oversimplifies the issues involved in language, communication, and learning. Research needs to stop construing every-day and school mathematical practices as a dichotomous distinction (Gutiérrez et al., 2010; Moschkovich, 2007, 2010; Schleppegrell, 2010). A theoretical framing of everyday and academic practices (or spontaneous and scientific concepts) as dichotomous is not consistent with current interpretations of these Vygotskian constructs (e.g., O'Connor, 1999) that assume every-day and academic practices are intertwined and dialectically connected. Classroom discourse is a hybrid of academic and everyday discourses and multiple registers co-exist in mathematics classrooms. Most importantly for supporting the success of students in classrooms, academic discourse needs to build on and link with the language varieties students bring from their home communities. Therefore, everyday practices should not be seen as obstacles to participation in academic mathematical discourse, but as resources to build on as students engage in the formal mathematical practices that are the focus of classroom instruction.

For example, the ambiguity and multiplicity of meanings in everyday language should be recognized and treated not as a failure to be mathematically precise but as fundamental to making sense of mathematical meanings and to learning mathematics with understanding. Ambiguity and vagueness have been reported as common in mathematical conversations and have been documented as resources in teaching and learning mathematics (e.g., Barwell, 2005; Barwell, Leung, Morgan, & Street, 2005; O'Halloran, 2000; Rowland, 1999). Even definitions are not a monolithic mathematical practice, since they are presented differently in lower level textbooks—as static and absolute facts to be accepted—while in journal articles they are presented as dynamic, evolving, and open to decisions by the mathematician. Neither should textbooks be seen as homogeneous. Higher-level textbooks are more like journal articles in allowing for more uncertainty and evolving meaning than lower level textbooks (Morgan, 2004), evidence that there are multiple approaches to the issue of precision, even in mathematical texts.

5.3 Kinds of Communication and Talk

Another important distinction is between different kinds of talk. Not all talk, even in classrooms, is the same, and this distinction can serve to clarify the setting, purposes, and genres involved in particular interactions. Below I summarize some distinctions among different types of communication.

Brenner (1994) provides useful distinctions among different kinds of communication in mathematics classrooms labeling them communication about mathematics, in mathematics, and with mathematics:

> Communication About Mathematics entails the need for individuals to describe problem solving processes and their own thoughts about these processes …. Communication In Mathematics means using the language and symbols of mathematical conventions … Communication With mathematics refers to the uses of mathematics which empower students by enabling them to deal with meaningful problems. (p. 241)

Another example is Barnes's (1992) distinction between exploratory talk and presentations (Barnes & Todd, 1995). Exploratory talk is "usually marked by frequent hesitations, re-phrasings, false starts, and changes of direction … such exploratory talk is one means by which the assimilation and accommodation of new knowledge to the old is carried out (p. 28)." Exploratory talk is especially important in small groups (Barnes & Todd, 1995).

Bunch (2014) makes a similar distinction between what he calls "the language of ideas" and "the language of presentations." Similarly, Herbel-Eisenmann, Steele, and Cirillo (2013) distinguish between what they call "communication contexts" and provide the characteristics of typical texts or types of language associated with each context. Communication contexts include working in a small group (language of interaction), reporting out to a whole class (language of recounting experience), a student writing a solution (language of generalizing), and a written description in mathematics textbook (the formal mathematics register).

The distinction among types of utterances is also important to consider. Analyzing teacher "questions" during classroom lessons is a typical research focus for novice researchers. If such analysis is not informed by language theories, it can result in a superficial definition of what constitutes a question, for example using only rising intonation to code for questions. Instead, Systemic Functional Linguistics provides a complex framework to analyze multiple forms of "questions" that can be applied to mathematics settings. SFL can provides a complex way to frame speech function and mood, so that utterances can be categorized as statements, questions, commands, or offers/requests. For example, Schleppegrell (2012) describes questions as follows:

> Typical: Interrogative: *What's the answer to number 2?*
>
> But also sometimes declarative: *I'm wondering if anyone has the answer to number 2.*
>
> Or imperative: *Raise your hand if you have the answer to number 2.* (p. 120)

6 Methodological Issues in Designing Research

I use the term "methodology" to refer to theory and methods together. I assume that methodology includes the underlying theoretical assumptions about cognition and learning: what cognition and learning are; when and where cognition and learning occur; and how to document, describe, and explain these phenomena. The issues raised in this section pertain not only to the methods one uses to examine language and mathematics learning but, more fundamentally, to how we theoretically frame and conceive of both mathematical activity and language.

When designing research on language and learning mathematics, it is important to consider what data to collect, which tools to use, and how. The design of data collection should consider and build on the relevant instruments used in previous research literature. It seems especially important to consider *how* we use video data to examine language, mathematical activity, and mathematics learning, particularly for evaluative analyses of student activity. It is a common experience when analyzing video data to focus on what a student is doing wrong rather than on what a student is doing well or how a learner is constructing meaning. Because video slows action down, participants on videotape may seem both less and more competent than in real time. As we watch video we have more time to notice how participants misspeak or make mistakes than we would have if we were observing in real time, thus making them appear *less* competent. When looking at video data, it is especially important to not equate a participant's linguistic competence with competence in mathematical reasoning. On the other hand, as we watch video we also have more time to notice and really think about what participants said and did, potentially making them look *more* competent than in real time. Thus, video data also opens up the possibility to document student competence in mathematical reasoning.

Transcription and transcript quality are theory laden (Ochs, 1979; Poland, 2002). Researchers make many decisions about transcripts that are based on their theoretical framework and on the particular research questions for a study. Decisions regarding what to include in transcripts and which transcript conventions to use are informed by theory. Whether a transcript will include gestures, emotions, inscriptions, body posture, and description of the scene (Hall, 2000; McDermott, Gospodinoff, & Aron, 1978; Poland, 2002) will depend on whether these aspects of activity are relevant or not to the particular research questions. Similarly, selecting transcript conventions and deciding whether overlapping utterances, intonation, and pauses are included or not in a transcript depends on whether these aspects are relevant to the research questions and analysis that will be carried out with the transcript and video. Lastly, whether and how different aspects of activity are relevant (or not) to the research questions depends on the theoretical framework.

7 Closing: Why These Issues Matter to Me

My goals in this chapter were to describe resources, challenges, and methodological issues to consider when designing research on language and mathematics learning based on over 25 years of research. As a researcher in mathematics education, I bring the lenses of the learning sciences and the field of mathematics education to these reflections. Because my own research focuses on mathematical thinking, learning, and discourse, both in monolingual and bilingual settings, I have had to read across several sets of research literature. In doing this inter- and cross-disciplinary work, I found that, while I remained grounded in my own field, I was forced to use perspectives from fields in which I had little formal training, such as bilingualism and second language acquisition.

My personal experiences of learning another language as a young child, being an immigrant, and becoming bilingual as an adolescent sparked my curiosity about bilingualism and second language acquisition. My commitment to improving the education of learners who are from non-dominant groups has provided my motivation and has sustained my dedication to tackling these issues. To develop the necessary grounding to conduct this research, I read across disciplines and fields. I found that I faced several challenges and recurring issues, but I also encountered useful resources. In this short piece, I only used references published in English (although I read work in other languages), mostly from the United States (because that is where my research setting is located), and from the United Kingdom. Elsewhere (Moschkovich, 2010), I have discussed in detail how to carefully use research carried out in geographic settings with student populations other than the target population for a particular study.

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Part II
Learner Focused Research

Making Student Explanations Relevant in Whole Class Discussion

Jenni Ingram, Nick Andrews and Andrea Pitt

Abstract Students explaining their mathematics is vital to the teaching and learning of mathematics, yet we know little about how to enable and support students to explain in whole class discussions beyond teachers asking particular questions. In this chapter we use a conversation analytic approach to explore the interactional structures that make student explanations relevant. Through a detailed examination of interactions where a student explanation occurs, three distinct structures are identified where a student explanation is perceived to be relevant. Our focus in the analysis is the social actions students themselves do in their explanations to display their interpretation of the interaction as requiring an explanation and constraining the type of explanation. However, these structures also offer ways that teachers can use the structure of interaction to encourage students to offer explanations in their responses.

1 Introduction

Students' explanations can be used by teachers to both monitor and respond to students' mathematical thinking and adapt their teaching in light of this (Franke, Fennema, & Carpenter, 1997). Describing, explaining and justifying can also help students develop their mathematical understanding (Rogoff, 1991; Sidney, Hattikudur, & Alibali, 2015). The act of students providing explanations for their answers has also been shown to be positively related to achievement outcomes (Webb & Palincsar, 1996). New mathematics curricula promote increasing the

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© Springer International Publishing AG, part of Springer Nature 2018

J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_5

lucidity of students' explanations, broadening their mathematical vocabulary, and more generally developing and communicating their mathematical understanding. However, most research into student explanations has either focused on the questions and prompts that teachers can ask so as to develop students' reasoning (e.g., Franke et al., 2009) or on the nature of the mathematical tasks that students engage with in their mathematics lessons. In these studies the teacher is explicitly inviting an explanation and the focus of the research has been specifically on the actions of the teacher. Yet student explanations are given by students, and do not always immediately follow an explicit initiation by a teacher.

In this paper we take a conversation analytic (CA) approach in order to examine the sequences of interaction during whole class discussions in which a student explanation occurs. Using the CA ideas of conditional relevance and preference we examine the structures within these interactions that mean that a student explanation is (conditionally) relevant. Other research analysing classroom interaction has focused either on categorising the types of explanations that students offer (Drageset, 2015) or on how teachers respond in the turn that follows (Franke et al., 2009; Lee, 2007). Here, we focus on the interactions that lead up to a student explanation that make a student explanation relevant during whole class interactions in mathematics lessons. This shifts the emphasis away from what teachers have done to generate student explanations. Instead the focus is on how students themselves use explanations to display their interpretation of the interaction through how they construct their responses. This approach emphasises the social action of justification and explanation, rather than a cognitive approach focusing on what students know or understand.

2 The Structure of Classroom Interaction

The most common structure in pedagogic interactions is the well-known three-part IRF (Initiation-Response-Feedback) (Sinclair & Coulthard, 1975; cf., IRE, Mehan, 1979, and "triadic dialogue", Lemke, 1985) sequence. This structure continues to dominate many classrooms and according to Wells (1993) "if there is one finding on which students of classroom discourse agreed, it must be the ubiquity of the three-part exchange structure" (p. 1). Numerous researchers (e.g., Nystrand & Gamoran, 1991) have criticised this structure as a teacher-dominant practice and question its role in meaningful student participation. Others, however, consider how the IRF sequence can be used as a teaching tool to guide a class of students toward the common goal of learning (Lee, 2007; Mercer, 1992; Nassaji & Wells, 2000).

IRF is not, however, the only interaction that takes place in the classroom (Cazden, 2001) and neither is it a single sequence type (Waring, 2009). The teacher "initiation" turns of IRFs (Mercer & Littleton, 2007; Milani, 2012) and the students' response turns (Drageset, 2015; Franke et al., 2009) carry out different types of actions, as do the third turns from the teacher which may launch a range of teaching activities (Lee, 2007; Nassaji & Wells, 2000). Whereas most of these studies focus on the categorisation of each turn in isolation, a CA approach focuses on the sequence of turns and the reflexive relationship between them. In particular, how participation in the IRF structure displays students' and teacher's actions and their interpretation of the turns that have gone before and constrain those that follow. The focus here is on how the students' response turns display the ways in which they understand the interactional context and act upon the turns that came before it. Specifically, how they use explanations to display their understanding of what their response needed to include. It is not about how these turns can be characterised with reference to categorical constructs. Although the analysis is not restricted to student explanations that arise within an IRF structure, the IRF categorisation does serve to simplify the structure of the interaction. Spoken interaction, and classroom interaction specifically, is inherently messy. Responses do not always directly follow the initiation, they can take several turns to be formed, and in classrooms responses can come from multiple speakers as well as single students. CA takes into account this messiness by focusing on the multiple actions performed by each turn and the sequential context within which the turns occur. In other words, it considers what the explanation is doing by being produced in that particular turn and in that particular way, in relation to the turns that have come before and those that might follow.

Whilst the ubiquitous nature of the IRF sequence alongside other underlying interactional structures imply some stability and reliability in classroom interaction, what teachers and students do in their turn positions is not predictable because it is contingent upon the immediate sequential context in which it occurs. Whilst it might be possible to predict that a student's response to a teacher's question will be an answer, it is not possible to predict the nature or form of that answer. Any predictability comes from the sequence of turns within which the turns occur, not from their categorisation as an initiation, response, or feedback move. The sequential context encompasses all those turns that influence the design and structure of the turns being considered.

In this chapter, we take a very broad definition of explanation as outlined below. In particular, the term 'student explanation' includes both turns that explain something and turns where there is disagreement between speakers. In ordinary conversation an explanation is given to address an assumed knowledge deficit in the other participants in the interaction (Antaki, 1994) and a distinction is often made between explanations and arguments (Quasthoff, Heller, & Morek, 2017). This distinction is not necessarily easy to make in the case of the classroom. Students can offer explanations to display their own knowledge without the requirement that there is a knowledge deficit in the other participants, for example the teacher. Explanations can also be given for why the student has given a particular answer, as well as for why a particular answer is valid. Furthermore, the distinction between an explanation and an argument can often only be made as the sequence unfolds as the nature of the disagreement that is the marker of an argument is not always immediately apparent in the interaction.

3 Methods

To illustrate the different sequential contexts within which student explanations are made relevant we draw upon data from a corpus of transcripts from mathematics lessons in the UK with secondary aged students (11–18 years old) which were products of several different studies. Each study involved the video recording of naturally occurring mathematics lessons but the overall aims of the studies and how the data was to be used varied between the projects. The lessons were naturally occurring in that there was no specific intervention or direction as to what should be taught or how it should be taught. In each case the teacher chose which lesson(s) with which class would be video recorded. The lessons came from teachers with a range of teaching experiences: some have taught for only 2 years whilst another has taught for over 30 years. The schools involved varied from state comprehensive schools serving areas with high levels of social deprivation, to a fee-paying selective boarding school. One aim of the current analysis was to look at commonalities in the structure of interactions around students' turns that include explanations across this diverse range of mathematics classrooms. All whole class interactions from these videos were transcribed using Jefferson (1984). In the extracts below the transcription has been simplified for ease of reading and the notation used is given at the end of this chapter.

The analysis takes advantage of conversation analytic methods to examine and specify the local contingencies that surround a student's explanation in order to identify the features of talk that make an explanation relevant. The approach is inductive with the focus of the analysis arising from 'unmotivated looking'. Initially turns where students were 'doing' explaining were identified, and these were then examined within their sequential context. A key principle of CA is that interactions cannot be adequately understood except by consideration of the sequential context in which they occur. Participants design their contributions to an interaction in light of the local context, particularly the other participants and the turns that have occurred before. These contributions then form part of the context for subsequent turns. These ideas emphasise the ethnomethodological roots of CA in that the focus is on how the participants themselves interpret the interactions, rather than the researcher. Hence, the focus of the analysis are those sequential contexts where students interpret the interaction as making an explanation relevant by offering (or attempting to offer) this explanation.

This analysis draws upon two key ideas from CA research, *conditional relevance* and *preference*. These two ideas describe the structure of the interactions rather than their usual psychological use. Within the IRF framework the response turn is *conditionally relevant* after the teacher has taken the initiation turn in that, following the initiation, a response is expected, or an account for why it cannot be given (Schegloff, 2007). Similarly, the third turn is conditionally relevant after the student has given the response turn. This is another way in which interactional structures in classrooms are different from those in ordinary conversation. Whilst a response is always conditionally relevant after an initiation, only in a teaching and learning interaction like those in classrooms is a third turn conditionally relevant following the response. So in exploring what makes a student explanation relevant we are examining the interactional structures within which the student displays this relevance in the way in which they construct their turn. In particular we are seeking whether there are features of the interactional context that students treat as making some form of explanation in their turn at talk as relevant.

Preference refers to the idea of affiliative and disaffiliative social actions, rather than the liking or disliking of the action. 'There is a "bias" intrinsic to many aspects of the organisation of talk which is generally favourable to the maintenance of bonds of solidarity between actors and which promotes the avoidance of conflict' (Heritage, 1984, p. 265). Certain responses to a teacher's initiation are preferred over others and these responses are usually affiliative and contribute to moving forward the interaction such as along a coherent and/or logical line of reasoning consistent with content-related goals of the lesson. A preferred response in this sense is one that is 'noticeably absent' when it is not given (Bilmes, 1988), principally because without such a response there is discontinuity in the line of reasoning that is being co-constructed by the teacher and students. In this sense, a preferred response could include errors or misconceptions if the teacher treats them as such by using them to move the learning towards their goals. But in classrooms where there are strongly established norms of turn taking, a secondary meaning of preference relates to whether the positioning of a response is consistent with the normative structure of talk in that classroom. In this case, the absence of a preferred response would be notable by an extended silence or *lapse* (Ingram & Elliott, 2014) or through a different student to the one the response was requested from offering a response. However, students do not always give a preferred response (or cannot) but the level of disaffiliation can be minimised if the dispreferred response includes an account or explanation. As with conditional relevance, we also examine interactional structures for features that students treat as signals that to include some form of explanation in their turn at talk is preferred, in the sense of progressing the conversation.

A defining challenge to this approach is identifying when student turns include explanations. Traditional categorisations, such as that offered by Drageset (2015), focus on the content of explanations but also fail to recognise unsuccessful attempts by students to offer an explanation, for example, when a student just says 'because…'. Explanations can also be formed over several turns and therefore categorising individual turns would also miss contexts where students perceive an explanation to be relevant. Therefore we took a broad approach to identifying sequences of turns that included student explanations. Initially we included all sequences where a student turn included a marker that might indicate an explanation, such as *because*, '*cause*, *therefore*, and *so* (the marker *hence* did not occur in the data). We also included all sequences where a student turn included more than two words to ensure that we did not miss explanations that did not use the traditional markers. The sequences of turns collected could include more than one student explanation, from one student or from several students due to the broad sense of interactional context used by CA. The analysis then progressed to identifying the interactional structures that made that student explanation(s) relevant which occurred in the lessons of at least two different teachers.

4 Findings

Three distinct interactional structures were identified within which a student included some form of explanation in a turn of talk. The first of these is where the student treats the teacher initiation turn as a direct signal that some form of explanation is relevant. Different forms of explanation might be offered within this structure as exemplified in due course. The second and third structures relate to where a student responds to an issue of preference by offering an explanation in their turn. In the second structure an explanation is used by the student to mitigate a dispreferred response they are giving which can arise in a range of interactional contexts, and in the third structure the explanation follows an indication that the original response given was not the preferred response. Each of these structures is discussed in turn below.

4.1 Teachers Explicitly Requiring an Explanation

In the data the vast majority of student explanations follow a teacher's initiation that makes an explanation explicitly conditionally relevant; often one that asks a why or how question. The asking of such questions, or a prompt that explicitly asks for a reason or an explanation, makes an explanation or reason conditionally relevant in the second turn. The student is expected to include a reason or explanation in their response.

1 T: okay Gabe. could you explain to me why (.) a quarter is, finding a quarter of something is (.) exactly the same as multiplying by point two five
2 S: er cause Tom said (.) er if you times a number say twelve by one it would equal the same. if you times it by more than one it would be more than that number. and if you times it by less than one but more than zero it would be er below the number ((inaudible)).

Extract 1—Teacher A

In Extract 1 the teacher explicitly asks for an explanation as well as asking why. Grammatically this is not a question but a request and an explanation is made relevant in the turn that follows. It is perhaps unsurprising that in these contexts students interpret the situation as needing an explanation and demonstrate this by including some form of explanation in their response. There is, however, considerable variation in the nature of explanations given by students. In Extract 2 the teacher also explicitly asks for an explanation but the response gives an explanation for why the student thought the answer was negative, not for why the answer was negative. The student has given a matched response in terms of the interactional structure, but not necessarily the response the teacher was anticipating.

1 T: why is it minus
2 S: because he just told me.

Extract 2—Teacher E

We do not consider the nature or quality of the student explanations in these scenarios here but this is considered elsewhere (Ingram, Andrews, & Pitt, 2016; Drageset, 2015). These contexts also give rise to failed attempts to give explanations, where the student has indicated that they see an explanation as relevant but does not actually give an explanation (e.g., 'because…'). In the first interactional structure the relevance of an explanation is explicit and the preference for some type of explanation over no explanation at all leads to a wide variety of explanations given, only some of which are mathematical in nature.

4.2 Giving a Dispreferred Response

Explanations or reasoning are also conditionally relevant when someone is giving a dispreferred response (Schegloff, 2007). This is particularly evident when a student is giving a response to a question that has already been responded to by another student earlier in the interaction. The current student needs to account for why their response is needed. By offering an explanation in their turn the student is showing that their response can be treated as dispreferred and are constructing their turn to mitigate this. This is the case both where the student has interpreted the context as needing to develop the previous student's responses in order to move the interaction forward, and also when they are offering a different, perhaps contradictory, response—again to contribute to the interactional trajectory of the lesson. By answering a question that has already been answered the student is treating this previous response as problematic in some way and therefore ensigns their turn to deal with the problem whilst minimising the disaffiliative nature of treating a peer's turn as problematic. The need for an explanation arises from a desire to mitigate the dispreferred nature of the response given and therefore this expectation for an account or explanation exists irrespective of whether the teacher prompts for it in a turn subsequent to the original student's response.

1 T: .. Michael do you think that's true or false
2 S1: true
3 S2: true 'cause there's only three outcomes to get so we-. ((transcript omitted))
4 S2: so it couldn't be unfair because its true its gotta be one third
5 T: ok does everybody else agree with that?
6 S3: no I don't. you've gotta say, imagine if it was like a good team and then a rubbish team. So they could lose. Like Accrington and Stanley.[1]

Extract 3—Teacher E

In Extract 3 the nominated student gives the answer to the question asked by the teacher in line 2. Another student, who was working with the first student on the task as a pair, then self-selects to repeat the answer and then add an explanation. The teacher has not explicitly asked for the explanation but the explanation adds to the answer given by the student's partner, S1. Another explanation is offered by a third student in line 6, which contrasts with the first two answers given. The teacher offers students the opportunity to offer a contrasting answer in line 5, but again does not explicitly ask for an explanation. In each case the explanation is part of a turn that dispreferred. In the first situation in line 3, the student had not been nominated to speak and had self-selected, and so is dispreferred because it is inconsistent with the normative structure of turn taking in this classroom. The explanation offers both an account for why they needed to speak in that they added to their partner's response and an explanation for why the statement is true. In the second situation the student was giving a contradictory answer to the answer offered by the two students before and is therefore not an affiliative action towards the other two students. Whilst this new answer contributes to the trajectory of the lesson, it marks out a discontinuity in the line of reasoning and consequently requires some form of account or justification.

Often teachers generated this expectation of an account through the posing of questions or tasks where different answers are either possible, such as an open question with multiple answers, or questions where a point of contention is likely to arise. These questions or tasks then offered more of an opportunity for students to give different responses to the same initiation. However, the account for giving a dispreferred response was not always accepted. In Extract 3 the omitted transcript includes an interaction between several students which indicates that the turn is treated as dispreferred by the other students, who tell each other to 'shut up' or similar.

Another situation where an explanation is used by a student to mitigate a dispreferred response arose where the student is giving a response that might alter the

[1]In the UK there is a well known television advert for milk where a young boy says that Ian Rush (famous footballer) said that if he didn't drink lots of milk he would only be good enough to play for Accrington Stanley to which the response was 'who are they', 'exactly'.

trajectory of the lesson, something that is usually tightly controlled by a teacher (Zemel & Koschmann, 2011). In Extract 4 the students are identifying a missing value when given an incomplete data set and the average of that data set. The discussion so far has focused on the mean.

44 T: so then, we know this. you know that those four add up to two hundred and seventy one, so then I suppose what you could do is say that two hundred and seventy one plus the maths mark, well has got to equal three hundred and fifty doesn't it. does that make sense? yeah Simon.
45 Ss: yeah
46 S: it could also be seventy one just because the question doesn't specify which average
47 T: hold on. We're going back to that, we will go back to that. Let's do this one. so, what is that number.

Extract 4—Teacher S

In line 46 the student is shifting the line of reasoning to another average, the median. His turn begins with an answer to the question that has generated the preceding discussions before accounting for where his answer has come from, and why it is another acceptable answer. This is similar to the situation above where the student is offering an explanation when they are speaking despite their partner being asked to speak. In both situations the right or authority of the teacher to control classroom interactions, whether that is the turn taking or the topic, is being challenged and the explanations serve to mitigate this challenge.

4.3 Students Continuing Their Turn

Finally, student explanations also become relevant when the expectation of them adding to their previous response arises, without the teacher necessarily explicitly asking for the explanation. This occurs when the turn to talk is returned to the student when it would normally be expected to return to the teacher (see McHoul (1978) for a detailed description of who can take turns when in classrooms). In the extract below there are two structural mechanisms that result in the turn returning to the student and them adding to their earlier response.

1 S: …I did nought point one was how long the blink would be
2 (0.9)
3 T: right
4 (0.8)

5 S: ti:mes (0.6) a hundred (0.8) because that's how fast it was going
6 T: so you did nought point one times a hundred
7 (1.8)
8 S: te:n (0.7) because I converted that into metres.

Extract 5—Teacher R

Firstly, there are several pauses in the dialogue of Extract 5 where the teacher would usually take the next turn: 0.9 s in line 2, 0.8 s in line 4, 0.8 s in line 5, 1.8 s in line 7 and 0.7 s in line 8. In each case, the pause indicates that the teacher is not going to take the turn as expected and therefore the turn returns to the student (see Ingram & Elliott, 2014 for a further explanation of why this occurs). This is supported further in line 3 where the teacher uses a continuation marker to indicate that he expects the student to continue. What the student does with the turn once it returns to them depends upon how they have interpreted the interaction that has led to this. The fact that the teacher has not taken the turn as expected indicates that there is some sort of trouble in the interaction and the silence is indicative of the absence of a preferred response. The student could interpret this as their answer not being acceptable in the sense of developing the line of reasoning, either because they have not included enough information, because there is an error in it, or because the teacher has not understood how they have phrased it. Most of these sources of trouble result in the student adding more clarification, such as an explanation, more information or some justification for their original response (Ingram & Elliott, 2016). In some cases it can result in students offering a different answer to the original question asked.

5 Conclusion

In this chapter we have identified three interactional structures that make student explanations *conditionally relevant* during whole class discussions. The first of these, where an explanation is explicitly asked for, has already been the focus of much research looking at the types of questions or prompts teachers can use that support students in giving their explanations. However, there is variation in the nature of the student explanations that follow as illustrated in Extracts 1 and 2. So whilst an explanation is conditionally relevant in the student's turn, there is not a simple correspondence between the type of question or prompt given by the teacher and the form of the student's explanation. An apparently straightforward 'why' question from the teacher makes logical (as in Extract 1 above), non-mathematical (as in Extract 2) and procedural explanations (what procedure was followed to get an answer) all conditionally relevant. Thus the student treats a 'why' question as a signal that some form of explanation is relevant, but only where the teacher initiation restricts these options does an explanation in the form of logical reasoning become conditionally relevant.

In the second interactional structure the student designs their turn as a dispreferred turn but mitigates the disaffiliative nature of the turn by adding an explanation. This turn could be dispreferred because the student did not have the right to speak but could also be a consequence of repeating an answer that has already been given and hence needing additional information to justify the repetition. In these cases the dispreference relates to the need to move the interaction forward in line with the goal of the lesson. The turn could also be dispreferred where it is offering a contrastive answer to another student (or the teacher). Here the dispreference relates to the affiliative action the turn is doing, and an account is needed to reduce the disaffiliative nature of the turn.

In the third structure using pauses or continuation markers to return the turn to the student results in the student needing to add to or alter their response. Here the disruption to the turn-taking structure of the IRF sequence indicates some sort of trouble with the student's response to the teacher's initiation that is preventing the final feedback turn by the teacher. The teacher is treating the student's response as dispreferred. In this case, the student treats the teacher refraining from taking the turn, as a signal to include an explanation in their next turn.

While our focus was on distinct interactional structures within which a student included some form of explanation in a turn of talk, each of these structures is manipulable by teachers within the structure of the IRF. Through the teachers' design of their initial turn, through the use of insertion IRF sequences prompting other students to build on or challenge the initial response to the initiation, or through the delaying (or omission) of the feedback turn. Future work in this project includes working with teachers in order to make use of these structures when supporting and encouraging student explanation and justification. Initial analysis of the current data indicates that there is wide variation in the nature of explanation following a teacher's explicit request for one. Explanations that occur in the other two structures are frequently about the mathematics, but include examples where the explanation is purely a description of the procedure used, and occasionally includes examples where the interaction breaks down. Further research is needed to examine how the qualities of the explanation varies within these structures, and how these qualities are a consequence of the preceding interactions.

Transcription Conventions

Taken from Jefferson (1984).

Convention	Name	Use
[text]	Brackets	Indicates the start and end points of overlapping speech
=	Equal sign	Indicates the break and subsequent continuation of a single utterance
(# of seconds)	Timed pause	A number in parentheses indicates the time, in seconds, of a pause in speech
(.)	Micropause	A brief pause, usually less than 0.2 s
.	Period	Indicates falling pitch or intonation
?	Question mark	Indicates rising pitch or intonation
,	Comma	Indicates a temporary rise or fall in intonation
-	Hyphen	Indicates an abrupt halt or interruption in utterance
°	Degree symbol	Indicates whisper, reduced volume, or quiet speech
Underline	Underlined text	Indicates the speaker is emphasizing or stressing the speech
:::	Colon(s)	Indicates prolongation of a sound
(text)	Parentheses	Speech which is unclear or in doubt in the transcript

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How Learners Communicate Their Mathematics Reasoning in Mathematics Discourse

Benadette Aineamani

Abstract This paper reports on qualitative research that draws on Gee's Discourse analysis to understand how learners communicate their mathematics reasoning in a multilingual classroom in South Africa. Data was collected in a Grade 11 class (14–16 years age group) of 25 learners in a township school in South Africa. Data was collected using classroom observations, and document analysis. The study showed that the way learners communicated their mathematics reasoning depended on the activities that were given by the textbook being used in the classroom, and the questions that the teacher asked during the lessons. From the findings of the study, mathematics classroom textbooks should be designed to enable learners communicate to their mathematics reasoning.

1 Introduction

Mathematics as a subject has its own Discourse. It has an accepted way of communicating mathematics. The Discourse includes a learner's point of view, beliefs, and thoughts about mathematics (Gee, 2005; Moschkovich, 2003). As learners take part in the mathematics Discourse, language is involved. The language that learners use should be constructed in such a way that they make a socially acceptable meaning of mathematics as a subject (Moschkovich, 2003). When communicating mathematics reasoning, language is used to express mathematics ideas and meanings, and in the process a mathematics register is developed (Pimm, 1991).

Prior to the study I report on in this chapter, I had conducted research on reasoning and communicating mathematically. The findings from my previous study showed that learners struggled to reason and communicate mathematically (Aineamani, 2010). Many learners in the study were unable to verbalise their reasoning clearly. They failed to speak and mean like mathematicians, which Pimm

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© Springer International Publishing AG, part of Springer Nature 2018
J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_6

(1991) identified as a goal for mathematics learning. These findings prompted me to conduct the study I am reporting on in this chapter. The focus of the study was to identify how learners communicate their mathematics reasoning. In this research, I focused on how communicating mathematically is influenced by what is legitimised within the mathematics discourse. In particular, I looked at how learners took part in a mathematics discourse in a school with a multilingual setting in South Africa. The three critical questions that guided the study were:

1. How do learners in a South African multilingual school interact with their teacher during a mathematics lesson?
2. How do these learners communicate their mathematics reasoning?
3. What language is used by the learners when they are interacting with their classmates, and when they are communicating their reasoning with the teacher?

2 Literature Review

2.1 Communicating Mathematics Reasoning

Over the years, mathematics teaching has moved from favouring a mechanical approach towards a view which encourages teachers to teach learners mathematics by emphasising problem solving, understanding, and communicating mathematically with others (McKenzie, 2001). Reforms in mathematics education invite teachers to provide a learning environment that encourages learners to connect mathematics ideas with the real world, explore mathematics ideas, and deepen their understanding (McKenzie, 2001). Stein, Grover, and Henningsen (1996) also contend that there is an increased emphasis in "doing mathematics." According to Stein et al. (1996), doing mathematics requires learners to be able to understand mathematics ideas so that they can take part in the process of mathematics thinking, and be able to do what mathematicians actually do. The essence of mathematics as a subject lies in the fact that all claims can be justified. Epistemologically, all knowledge that we hold should have a basis and we should be in a position to explain and justify the knowledge that we hold (Johnston, 2002). Once an individual is able to give an explanation of why something is the way it is, opportunities for developing a well based understanding of the knowledge are afforded to the individual. As such, it can be argued that the individual is better positioned to refer to such knowledge as his/her personal knowledge since s/he is able to justify it.

Once learners are able to apply the unique features of mathematics such as communicating their mathematics reasoning, they are then able to solve problems that require them to generalise, to apply abstract thinking and to also simplify. Therefore, mathematical reasoning is very useful because it enables learners to solve problems that they have not come across before by using justification and generalisation techniques in the process of answering a given question (Kilpatrick, Swafford, & Findell, 2001).

However, Sfard (2001) argues that finding ways to make the idea of learning mathematics with understanding work in classrooms is extremely difficult to achieve. In order for learners to be able to communicate mathematically with their teacher and peers, they must have developed a language of mathematics that enables them to express themselves (McKenzie, 2001). Most of the time, teachers expect their learners to be able to communicate their mathematics ideas effectively, but this may not be the case in some instances (McKenzie, 2001). Leaners need to be given an opportunity to develop appropriate ways of communicating mathematically. Bicknell (1999) argues that as learners take part in discussions amongst themselves, and with their teacher, they are provided with a chance to take part in a social interaction and as a result, their understanding is negotiated and developed.

Reasoning mathematically forms the foundation of mathematical understanding (McKenzie, 2001). Therefore, mathematical understanding depends on reasoning, which means that reasoning is very important for a learner to grow in mathematical knowledge (Muller & Maher, 2009). Once a learner is able to reason mathematically, s/he is able to apply the mathematical ideas to new situations and hence problem solving skills are developed (Muller & Maher, 2009). Sfard (2001) argues that placing communication at the centre of mathematics education is most likely to change the ways people think about the process of learning mathematics and about what is being learnt in mathematics classrooms. Communication is not simply an aid to thinking but it is a requirement for one to reason mathematically (Sfard, 2001).

In South African classrooms, the teacher is looked at as a source of the mathematics knowledge and so the learners wait for the teacher to decide for them what to do any time they are faced with a mathematics problem (Brijlall, 2008). This makes such learners dependent on their teacher. For learners to develop problem solving skills, they have to become independent and critical thinkers (Brijlall, 2008). Therefore, the issue of reasoning and communicating mathematically should be addressed and emphasised in mathematics classrooms so that learners are given the opportunity to become independent and critical thinkers. Once learners are given an opportunity to develop their reasoning, their attitude towards mathematics as a subject may change for the better (Brijlall, 2008).

2.2 Communicating Mathematics Reasoning in a Multilingual Classroom

Multilingualism in South Africa is complex. This is reflected in the classrooms whereby learners speak different languages (Brijlall, 2008). The learners in South African classrooms speak English, Afrikaans, isiZulu, seSotho, and siSwati among others. In such classrooms, reasoning and communicating mathematically might be affected by the language challenges. Reasoning and communicating mathematically may become problematic in a multilingual classroom as shown by the study conducted by Barton and Barton (2005) in a multilingual classroom in New Zealand. They found that, due to the fact that English was used the language of teaching and learning, students whose home language was not English had difficulties of understanding the vocabulary used in the mathematics discourse as a whole. They also found that some mathematics terms that are used in everyday contexts caused confusion for the students in the mathematics classroom. They found that the students who were not first language speakers of English had a 10–15% disadvantage due to language difficulties. The worst part of the challenge of language was that the students were not aware of their problem. Reasoning and communicating mathematically may become problematic in a multilingual classroom as shown by the study conducted by Barton and Barton (2005). Language is a tool that is required for one to think and communicate mathematically (Setati, 2005). Therefore language should not be underestimated in the process of teaching mathematics to learners.

In a mathematics classroom, learners are expected to write and talk mathematically (Moschkovich, 2003). As learners talk and write about mathematics, they are communicating mathematically (Pimm, 1991). In order to investigate how learners communicate their mathematical reasoning, both oral and written communication have to be considered. This is because some learners maybe able to orally communicate their reasoning and fail to write it down while others are able to write down their reasoning while not be able to orally communicate their reasoning.

The language used to communicate mathematically is also very important. Cleghorn and Rollinick (2002) carried out a study in which they found that learners participate more lively and freely when they are allowed to use their home languages to talk about mathematics than when required to use English. This highlights the importance of language when learners are required to communicate their mathematical reasoning.

3 Theoretical Perspective

The notion of communication in this study was informed by Gee (2005) who argues that language is situated. He, therefore, says that in order to study any language that is being used to communicate, we must consider more than the language. In other words, for one to study any language, one has to study the Discourse in which that particular language is used (Gee, 2005). Gee (2005, p. 36) distinguishes between Discourse with a capital "D" and discourse with a lower case "d". He defines Discourse, with a capital "D" as "ways with words, deeds and interactions, thoughts and feelings, objects and tools, times and places that allow us to enact and recognize different socially situated identities". Mathematics as a subject has its own Discourse and so learners in a mathematics classroom are expected to use ways, deeds, and interactions that are part of the mathematics Discourse. Gee defines discourse (with a lower case "d") as the actual language that is used in the Discourse. For example, mathematics discourse (with a lower case "d") refers to mathematics language, e.g. in mathematics reasoning it is the language that is used in proofs and mathematics conjectures that learners may formulate in the classroom. A mathematics discourse requires a learner to be able to reason and communicate mathematically since mathematics discourse is reasoned discourse (Moschkovich, 2003).

Gee (2005) discusses the idea of *social language* in order to show that language alone is not sufficient for one to participate in a given Discourse. Social languages are "what we learn and what we speak" (Gee, 2005, p. 38) in a given social setting. For every setting that an individual finds her/himself in, there is a different social language that one has to use in order to participate in that particular setting. Gee (2005) argues that there is a formal and informal setting. If one finds oneself in either a formal setting or an informal setting, one is expected to use a different social language (Gee, 2005). For example, when a learner is communicating with the teacher, he may use a different social language that is different from the one that the learner uses when s/he is communicating with peer(s) about the same idea because the learner is communicating within two different social settings, with the teacher, and with the peer. Pimm (1991) argues that there are informal and formal settings that learners find themselves while at school.

According to Pimm (1991), in a mathematics classroom learners learn to move from informal spoken languages which they use outside the classroom setting (informal setting) to a formal spoken or written activity which is viewed as a requirement for the learners to participate in the mathematics activities. Learners are also required to speak in a formal way in the mathematics classroom because the classroom is a formal setting (Pimm, 1991). For example, a learner may want to talk about variables in a mathematics classroom. This learner is expected to be explicit and say "the variable x" and not "letter x" because the word variable makes the language more formal in a mathematics classroom. The movement from informal communication to formal is not easy for the learners because they come to school fluent in communicating informally (Pimm, 1991).

Moschkovich (2003) also argues that for learners to take part in a mathematics Discourse, they have to move from an everyday way of talking to a more precise way of using mathematical language. For learners in multilingual classrooms the movement also includes moving between languages (Setati & Adler, 2000) and moving between cultures (Cleghorn & Rollinick, 2002; Zevenbergen, 2000). Cleghorn and Rollinick (2002) refer to the movement between the culture of the home and the culture of the school as border crossing. Second language learners have to make a border crossing as well as moving between their home languages and English (the language of teaching and learning) and between informal and formal mathematics language. Thus they have to navigate between numerous social languages.

4 Research Design and Methodology

The study was a qualitative study. Learners were observed in their natural setting of the classroom, without the researcher trying to manipulate the learners' behaviour during the lessons. The sample comprised one school located in a township school in South Africa. From the description by the teacher, the majority of the learners were from child headed families, and they struggled to cope with school work. There was a high rate of absenteeism which affected the performance of the school. Only one class of Grade 11 learners, aged 14–16, was selected to take part in the study. In order to answer my research questions, I observed how the learners were communicating their mathematics reasoning orally and in written texts. I observed the classroom continuously for two weeks and recorded four lessons that were each 60 min long. I transcribed the data and got people to translate for me the different languages that learners were using during the lessons. A total of six learners' mathematics note books were collected and analysed, with permission from teacher, and the learners themselves. The mathematics textbook used in the classroom by the teacher and the learners as reference was also analysed to find out the communication practices legitimised by the textbook.

I used the typological analysis method discussed by Hatch (2002) to analyse my data. Typological analysis is where data analysis is started by dividing the collected data into a set of categories that are based on predetermined typologies. The typologies were generated from the theory, common sense and objectives of the research. I used this method because the topics in mind are usually the logical places to start in the process of analysis (Hatch, 2002). Hatch (2002) argues that data analysis happens within the generated typological groupings (Hatch, 2002). Therefore, I analysed my data, including the transcribed portions of the video, and "divided it into elements based on predetermined elements" (Hatch, 2002, p. 152).

5 Findings from the Study

The findings from the study show that the teacher engaged the learners during the lesson but she did not probe learners to elaborate on their communication during the lesson. From the excerpt below, the learners were responding in short phrases and one word answers. The response from the teacher about the answers given by learners shows that she did not mind the learners answering using the short sentences and phrases.

Teacher: Write there, what do they say, they say find the mean, ne [okay], you can calculate from here, you don't have to transfer data from stem and leaf to, because already…

Nicole: They have given the scores

Teacher: Eeh people exercise eight comma ten, ne [okay], you have been given scores there as stem and leaf you don't have to re-write, right, already stem and leaf is giving you what, the…

Class: …The scores

Teacher: The scores from the smallest to the…

Class: …Highest

Teacher: Highest, do you understand, time management, hullo can I have your attention please, stem and leaf, your data already has been arranged in an ascending order okay, please time management it's also important because if you re-write things that are not necessary you won't finish the question okay, do you understand, so that data there is giving us stem and leaf, just calculate your mean, median, whatever is asked there, okay.

The findings from the study also show that learners communicated with peers using languages other than English, and they switched to English when communicating with the teacher. The teacher encouraged the learners to communicate their mathematics reasoning in English during the lesson. The teacher did not tell the learners to speak English but she expected the learners to follow her example of not using any other language other than English. The teacher did not probe learners for further explanation when they attempted to communicate their mathematics reasoning.

While some of the activities in the textbook used in the lessons required the learners to communicate their mathematics reasoning, most of the questions required learners to carry out procedural manipulations because the questions required the learners to recall a given formula or way of working out a given problem and reproduce it to answer the given question (Kilpatrick et al., 2001). Most of the questions in the textbook were in the format below.

Find the mode, median and mean of the following values: 1; 5; 7; 3; 5; 9; 5; 8; 10.

The example above demonstrates how the communication practices legitimised by the textbook were procedural because most of the questions and the activities in the textbook, as evident in the example above, did not require learners to give a justification for the solutions. After learners answered questions, the teacher did not ask the learners to explain their answers. Therefore, learners in this study communicated procedurally as required by the textbook. The excerpt below shows how one of the learners communicated her mathematics reasoning by talking about the procedure.

Teacher: Sixteen, okay and thirty five, okay, let's see you say half its sixteen, sixteen it means the first sixteen and the last sixteen, so let's talk about thirty two, thirty two, is it the even number or the odd number

Nicole: Even number

Teacher: So if it's an even number what do we do?

Nicole: We add the numbers and divide by two because we want to find the half of it.

Teacher: Ja [yes] you must take score number sixteen plus score number seventeen we add it together and divide by two, is it what you did, is it what you did

Nicole: Yes.

Within the mathematics reasoning Discourse, learners are required to provide justification for any response they give to a problem (Muller & Maher, 2009). However, the textbooks in my research were designed to teach students particular mathematics techniques and procedures rather than to help students develop thinking skills necessary for the learners to take part in the mathematics reasoning Discourse. The excerpt from the transcript above also shows that the Discourse legitimised in the classroom did not promote mathematics reasoning. From my experience of mathematics textbooks in South Africa and beyond, the majority of textbooks seem to be similar in this way.

6 Conclusions and Recommendations

Based on the research I have described here, and substantiated by my years of experience within mathematics education, I make the following recommendations—first, a recommendation about textbooks. Mathematics knowledge by its nature has a foundation in reasoning. Reasoning refers to the use of logical and coherent arguments to form conclusions, inferences, or judgments (Ross, 1997). Reasoning can also be defined as the process of drawing conclusions on the basis of evidence or stated assumptions. Mathematics relies on logic and it is through this logic that mathematics knowledge can be justified. Without reasoning, mathematicians would not be able to convince other people that their conclusions are true, and make sense (Muller & Maher, 2009). Ross (1997) argues that mathematics lies in proof, yet proof requires reasoning. Therefore, since mathematics as a discipline lies in proof, and since there is no way a proof can be constructed without reasoning, then reasoning is the foundation of mathematics, as argued earlier. Therefore, learners should be given an opportunity to explain and justify their arguments and answers within the mathematics discourse. In other words, learners should be given an opportunity to communicate their mathematics reasoning. The textbook used in the classroom, together with the teacher help in enabling or restricting learners to communicate their mathematics reasoning. If learners are not asked to communicate their mathematics reasoning, they do not communicate their reasoning as shown in the study.

Secondly, my research has implications for teachers. The questions a teacher asks help the learners to communicate their mathematics reasoning. The way learners communicate their mathematics reasoning depends on the activities that are given by the textbook being used in the classroom. If the textbook used in the classroom does not have activities and questions that promote mathematics reasoning, learners do not learn how to communicate their mathematics reasoning since they are not exposed to activities and questions which require them to communicate their mathematics reasoning (Stein et al., 1996). Mathematics textbooks should be designed to enable learners to communicate their mathematics reasoning. Asking open ended questions and questions that require learners to justify and give explanations to their answers should be included in the textbook. Therefore, in order for learners to communicate their mathematics reasoning, they should be probed or asked the right questions that enable them to communicate their mathematics reasoning.

Finally, my research has implications for assessment and curriculum. The assessment of how learners communicate their mathematics reasoning should have a basis in curriculum standards. If the curriculum states the level of mathematics reasoning which the learners at every grade must reach, then the teacher will have to probe the learners for higher reasoning. Therefore, assessment of the mathematics reasoning of learners should have a frame of reference so that the teacher is aware of the level to which the learners are expected to reach when communicating their mathematics reasoning at the relevant grade level. Without a frame of reference, the teacher may legitimise very low levels of mathematical reasoning that are not at the academic standards of the learners.

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Dealing with Function Word Problems: Identifying and Interpreting Verbal Representations

Carina Zindel

Abstract The intertwined conceptual and language demands of word problems for functional relationships can be challenging. A design research study containing 16 design experiments in a laboratory setting with ninth and tenth graders explored one typical challenge, recognizing the core of functional relationships in different representations. Adequately connecting verbal and symbolic representations requires identifying and interpreting the verbal representation by addressing the relevant facets. A qualitative analysis of students' solution processes shows different approaches to this task.

1 Introduction

According to Moschkovich, "studies should focus less on comparisons to monolinguals and report not only differences between monolinguals and bilinguals but also similarities" (Moschkovich, 2010, p. 11). Therefore, the aim of this study is to identify language demands all learners face, not only learners with the language of instruction as a second language. Language demands depend on the underlying role of language. The demand of reading word problem texts illustrates the communicative role of language (Abedi & Lord, 2001; Hirsch, 2003). However, tasks with high conceptual demands are more difficult than tasks with reading obstacles (Prediger, Wilhelm, Büchter, Gürsoy, & Benholz, 2015; Ufer, Reiss, & Mehringer, 2013). The epistemic role of language describes language as means of conceptual development (Heller & Morek, 2015).

Language proficiency is a factor with a strong connection to mathematics achievement, stronger than multilingualism, immigrant background, or socio-economic status (Heinze, Reiss, Rudolph-Albert, Herwartz-Emden, & Braun, 2009; Prediger, Wilhelm, Büchter, Gürsoy, & Benholz, 2015). This applies to dealing with word problems, which is difficult for many learners, especially (but not

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© Springer International Publishing AG, part of Springer Nature 2018
J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_7

only) second or additional language learners. This fact leads to the question, which linguistic means are important with respect to functional relationships and how language demands and conceptual demands are related (cf. Prediger & Zindel, 2017). Describing these demands in more detail requires further subject and topic-specific analysis of language demands. This specification allows the development of language and mathematics integrated teaching-learning material, which is a second aim of this study.

Studies dealing with the teaching and learning of the function concept often refer only to the symbolic, numeric and graphic representations (Leinhardt, Zaslavsky, & Stein, 1990; Moschkovich, Schoenfeld, & Arcavi, 1993; Romberg, Fennema, & Carpenter, 1993) or to the connection between verbal and graphic representations (Swan, 1985). However, the verbal representation is often not taken into account. Hence, in the larger research project (Zindel, in preparation) the roles of the verbal representation as learning goal and learning medium are taken into account and analysed with regard to potential obstacles and hypothetical learning trajectories. One important issue concerning dealing with function word problems is the question: How do learners interpret verbal representations of functional relationships? The interpretation here includes identifying the respective verbal representation in the situation described.

2 Theoretical Background and Research Questions

2.1 Dealing with Function Word Problems

Word problems can be posed in many different ways. For this reason, this paper differentiates two parts of word problems: the *situation* and the *question*. The situation can include verbal and/or symbolic representations for example. The question asks for information about the situation.

With this distinction, solving word problems requires the following steps (not necessarily in this order): (1) identifying and interpreting the *functional relationship* in the situation, (2) identifying the *involved quantities* in their roles as *given quantity* and *unknown quantity*. And, (3) connecting the information about the *independent variable* and *dependent variable* with the value of the identified given quantity enables calculating the unknown quantity and thereby solving the word problem.

Adequately dealing with function word problems (word problems for functional relationships) requires interpreting the *situation* (here including verbal and symbolic representations) with regard to the *questions*. A question contains a given quantity and an unknown quantity. In order to calculate the unknown quantity one needs to identify the given quantity and the unknown quantity within the question as well as the independent variable and the dependent variable in the functional relationship described in the situation (independent variable, dependent variable).

FlatWatch2.0

The streaming provider FlatWatch2.0 offers renting an unlimited number of films for a fixed-price per month. Additionally, there is the possibility to buy films permanently. The functional equation $f(x) = 5\,x + 4$ states the price in one month depending on the number of films bought.

"How many films did you buy when you have to pay 59€?"

Fig. 1 Example for a function word problem

Therefore, it might be necessary to connect the information of different representations in the situation.

Figure 1 illustrates these demands. The task consists of an offer of a streaming provider and both, a symbolic and a verbal representation describing the functional relationship. In order to answer questions such as "How many films did you buy when you have to pay 59€?" one needs to know the meaning of the variables x and f(x). Having identified the number of films bought as unknown quantity in the question and independent variable in the situation, and the price in one month as the given quantity in the question and the dependent variable in the situation, one can calculate the x by plugging in the 59€ for f(x). However, identifying the roles of the involved quantities in the situations might be challenging for learners. In order to do this, one needs to connect this symbolic representation with the information given in the verbal representation.

2.2 *Connecting Representations of Functional Relationships*

Connecting representations (verbal, symbolical, numeric and graphic) has often been described as an important activity for a deep understanding of concepts like function (e.g. Duval, 2006; Swan, 1985). Typical challenges in connecting representations for functions have been reconstructed (e.g. Swan, 1985). These challenges are often specific for certain types of functions or certain representations. For example, concerning the connection graphical and algebraic representations, Moschkovich (1998) describes that students often mix up the x-intercept with the constants in linear equations. One reason might be missing conceptual understanding, involving "understanding the connections between … representations, knowing which objects are relevant in each representation, and knowing which objects are dependent or independent." (Moschkovich, 1998, p. 212). So, on the one hand, connecting representations is an important tool for concept formation processes (Duval, 2006). On the other hand, it is challenging for many learners (Leinhardt et al., 1990; Niss, 2014). One problem could be that the conceptual core common to all representations of the function concept remains implicit (Niss, 2014).

The core of the function concept can be conceptualized by the *facet model* (cf. Prediger & Zindel, 2017; Zindel, in preparation). This model is based on the cognitive psychological assumption that concept knowledge is a mental network consisting of many so called "comprehension elements" (Drollinger-Vetter, 2011) we call *facets* of the concept. Facets can be connected and compacted into denser concepts (Aebli, 1981; cf. Glade & Prediger, 2017). Depending on the situation learners need to be able to flexibly compact and unfold the facets (Drollinger-Vetter, 2011).

The model printed in Fig. 2 summarizes the facets of the core of the function concept that are relevant in all representations and for all function types (Prediger & Zindel, 2017). In the following, the facets are briefly explained and exemplified in a hypothetical process of solving one function word problem. The facets in the boxes of the model are indicated by ‖…‖ in the text. The edges describe the required connections between the facets, which can be interpreted either as processes of unfolding (when reading top-down) or as processes of compacting (when reading bottom-up). Reading the model bottom-up, it is important to know that there are (mostly two) ‖involved quantities‖ in order to identify ‖quantity I‖ and ‖quantity II‖. Combined with the facet ‖quantities vary‖ these quantities can be seen as ‖varying quantity I‖ and ‖varying quantity II‖. Thompson (2011) emphasises the relevance of this perspective by the claim for "quantitative reasoning". This requires seeing the "invariant relationship between two quantities whose values vary" (Thompson, 2011, p. 46). Combined with the facet ‖direction of dependency‖ these varying quantities can be addressed as ‖independent variable‖ and ‖dependent variable‖. Identifying the ‖independent variable‖ and the ‖dependent variable‖ has often been described as challenging for learners (e.g. Moschkovich, 1998).

Adequately dealing with function word problems requires adequately connecting the verbal and symbolic representations. In order to successfully connect representations, the facets of the model need to be flexibly unfolded or compacted. Again, the example of FlatWatch2.0 (Fig. 1) illustrates this demand. In order to answer questions like "How many films did you buy when you have to pay 59€?" one needs to know the meaning of the variables x and f(x). Therefore, the identification and interpretation of the linguistic means *depending on* in the verbal representation is necessary. The interpretation includes recognizing both ‖involved quantities‖ *price in one month* and *number of bought films*. Additionally, you need to know that these ‖quantities vary‖ so that you are allowed to plug in values for them. Compacting these facets leads to addressing ‖quantity I‖ and ‖quantity II‖ as

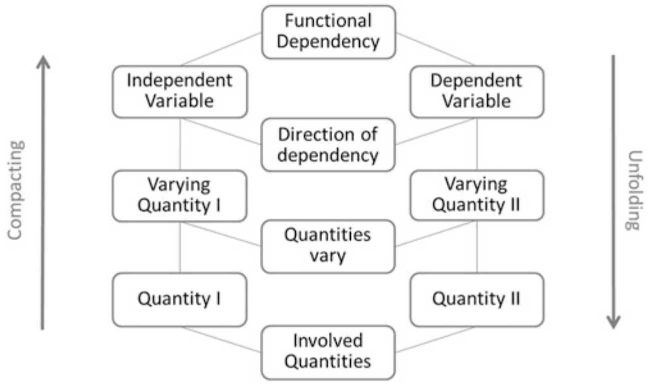

Fig. 2 Facet model of the core of the function concept

||varying quantity I|| and ||varying quantity II||. Finally, you need to decide whether to plug in the given 59€ for x or f(x). Therefore, recognizing the ||direction of dependency|| is important. The verbal representation "*price in one month depending on the number of films bought*" includes the information that "the price in one month" is the dependent variable and "the number of films bought" is the independent variable. Interpreting this linguistic means with regard to the symbolic representation leads to the information that you have to plug in the 59€ for f(x).

2.3 Language Demands When Dealing with Function Word Problems

Many studies identify language demands by analysing textbooks and curricula (Bailey, Butler, Stevens, & Lord, 2007; Thürmann, Vollmer, & Pieper, 2010). The phrases for expressing functional relationships are very diverse in German. Besides this verbal diversity, the German language allows many variations in grammar without changing the semantic content (e.g. changing the order of subject and object in a sentence is possible without changing the meaning of the sentence). An excerpt of this diversity is shown in Table 1.

Language demands for dealing with function word problems can be subdivided into *receptive* and *productive* demands (cf. Prediger & Zindel, 2017).

Receptive demands are (as already mentioned above) the identification of verbal representations in the situation respectively of the linguistic means within the verbal representation. Each representation is based on the use of a semiotic register (Duval, 2000). The verbal representation is based on a verbal semiotic register. With respect to (multilingual or monolingual) learners, Prediger, Clarkson, & Bose (2016) differentiate the verbal semiotic register into the everyday register, the school academic and technical register. Considering the large variety of phrases for expressing

Table 1 Receptive demand: variety of phrases for expressing functional relationships

f(A) = B	Active sentence structure	Passive sentence structure
Dependency B of A	*The function indicates B in dependency of A* Die Funktion gibt B in Abhängigkeit von A an Die Funktion gibt das von A abhängige B an Die Funktion gibt B an, das von A abhängig ist Die Funktion gibt B an, das von A abhängt	*In the function, B is given in dependency of A* B wird in Abhängigkeit von A angegeben Es wird das von A abhängige B angegeben Es wird B angegeben, das von A abhängig ist Es wird B angegeben, das von A abhängt
Assignment A → B	*The function assigns each A to a B* Die Funktion ordnet jedem A ein B zu Die Funktion ordnet ein B zu jedem A zu	*Each A is assigned to a B* Jedem A wird ein B zugeordnet Ein B wird jedem A zugeordnet
Implicit description by prepositions	*The function gives a B for* [*to*] *each A* Die Funktion gibt **für** jedes A ein B an Die Funktion gibt **zu** jedem A ein B an	*For* [*to*] *each A, a B is given* Es wird **für** jedes A ein B angegeben Es wird **zu** jedem A ein B angegeben

functional relationships in the German school academic and technical register (Table 1), this appears to be important for all students, not only for language learners.

Productive demands occur whenever students verbalize functional relationships. This activity is especially important when interpreting the verbal representation regarding the symbolic representation. Interpreting requires addressing facets of the model. Representations of functional relationships include the information about the given functional relationship in a compacted way. However, for processes of unfolding and (re-) compacting, it could be helpful or even necessary to verbalize the facets on the lower levels. This requires activating further linguistic means.

2.4 Research Question

This study addressed the following research question: Which facets do learners address in order to identify and interpret verbal representations of functional relationships when assigning questions to situations?

3 Research Design

In the larger research project, within the research program of topic-specific Didactical Design Research (cf. Prediger & Zwetzschler, 2013), three design experiment cycles were conducted that intertwine the research activities of specifying the conceptual and language demands, designing and refining a teaching-learning arrangement and deepening the empirical insights into students' learning processes. In total, 16 design experiments were conducted with pairs of learners with heterogeneous language backgrounds (in each case 1–3 sessions). Additionally, design experiments in three classrooms were conducted in a fourth design experiment cycle.

The following case studies of four pairs of learners (grade 9, most of them 15 years old) come from the third design experiment cycle. The students were already familiar with linear functions. An overarching design principle in the designed teaching-learning arrangement is *problematizing the core of functional relationships*. Several design elements in the teaching-learning arrangement implement this principle. One of these design elements is dealing with *varying phrases*. These phrases vary in the facets of the model (Fig. 2) in order to raise students' awareness of these facets (cf. Prediger & Zindel, 2017). Another design element implementing this principle is the following type of task: *Assigning questions to situations* (Fig. 3).

Solving function word problems is a complex task, which can be analysed as involving in several steps. A first step in solving such problems might be assigning questions to situations, in this case to the varying offers of the two streaming providers FlatWatch2.0 or RipOff. In order to assign the questions to the situations adequately, it is necessary to identify the verbal representation in the word problem text (e.g. "price in one month depending on the number of bought films") and to interpret the verbal representation regarding the ‖involved quantities‖. These

FlatWatch2.0	**RipOff**
The streaming provider FlatWatch2.0 offers renting an unlimited number of films for a fixed-price per month. Additionally, there is the possibility to buy films permanently. The functional equation $f(x) = 5\,x + 4$ states the price in one month depending on the number of films bought.	Sina found another streaming provider. Here the following functional equation states the total price depending on the number of months: $f(x) = 25\,x + 9.99$
How many films did Sina buy when she has to pay 59 Euro? What is Sina going to pay in one month if she has bought 5 films?	What is the total price after 6 months? After how many months does Sina pay 84.99 Euro?

Fig. 3 Activity: assigning questions to situations

||involved quantities|| have to be the same in both, the situation and the assigned questions. So adequately assigning the questions to the situation requires addressing the ||involved quantities|| as facets on a lower level of the model. Section 4 shows students' approaches to solving this first step of the task. The second step would be to answer the question, which requires additional facets, especially the ||direction of dependency|| (cf. Zindel, in preparation). This aspect is not considered in this report.

The design experiments were videotaped, partly transcribed, and qualitatively analysed with respect to the research question. Facets of the core of the function concept being relevant for connecting representations were gathered in a category-developing qualitative procedure. The resulting empirically grounded model of the core of functional relationships allows the analysis to distinguish between different approaches that connect both representations.

4 Selected Results from the Analysis

The empirical results indicate that conceptual and language demands are related. Learners need support in order to be able to identify and interpret the relevant verbal representation in the situation. This study presents brief excerpts of four case studies (transcripts literally translated from German), illustrating different approaches by learners ranging from addressing individual (non-adequate) facets to adequately addressing facets of the core of the functional relationships.

4.1 Brief Summary of the Case of Alexandra and Tatjana

Alexandra and Tatjana (14 and 15 years old) adequately assigned the two questions "What is the total price after six months?" and "After how many months does Sina pay 84.55€?" to the RipOff situation. They also adequately assigned the two other questions to the FlatWatch2.0 situation. Next, the teacher asked for an explanation.

204	Tatjana	*[4s]* Well here *[points to the RipOff situation]* the functional equation states – um – the total price depending on the number of months. And –
205	Teacher	(…) *[approvingly]* Uh-huh.
206	Tatjana	And here *[points to the questions "What is the total price after six months?" and "After how many months does Sina pay 84.99€"]* it is the total price after six months and after – so the total – the number of months.

In line 204, Tatjana addressed the ||functional relationship|| in the RipOff situation by repeating that the functional equation states "the total price depending on the number of months." She identified the relevant verbal representation in the situation and addressed the ||functional relationship|| and thereby the ||independent variable|| and the ||dependent variable|| in a compacted way as she adopted the phrase from the verbal representation in her verbal explanation. In line 206, she identified the same ||involved quantities|| in both questions, too. After being prompted by the teacher, she justified the fit of the FlatWatch2.0 situation to the two other questions.

208	Tatjana	Yes here *[points to the FlatWatch2.0 situation]* it is like (…) – well – like the opposite. That – um – you can also buy films. And here it also says *[points to the question "How many films did Sina buy when she has to pay 59€?"]* how many films did Sina buy.
209	Teacher	*[approvingly]* Uh-huh.
210	Tatjana	When she paid that much and *[4s]* like here *[points to the question "What is Sina going to pay in one month if she has bought 5 films?"]* how much she paid in one month, when she bought five films.

Similar to her first explanation, she explicitly referred on the one hand to the situation and on the other hand to the question. In line 208, she first focused on the FlatWatch2.0 situation and noticed, "you can also buy films". Thereby she addressed the number of films bought as one of the ||involved quantities|| in the situation. Then she referred to the question "How many films did Sina buy when she has to pay 59€?" and noted that it contains the same ||involved quantity|| by reproducing the part of the question that makes the ||unknown quantity|| explicit. She continued her explanation by referring to the ||given quantity|| in the question she had just assigned (#210). In line 210, she additionally argued that the second question ("What is Sina going to pay in one month if she has bought 5 films?") contains the same ||involved quantity|| as well. Therefore, she reformulated the question and addressed the ||given quantity|| and the ||unknown quantity|| in that question.

In this way, Tatjana identified and interpreted the relevant verbal representation with regard to the symbolic representation. This allowed her to assign the questions to the situations adequately.

4.2 Brief Summary of the Case of Fynn and Svenja

In contrast, Fynn and Svenja (both 15 years old) did not focus on the verbal representation of the functional relationship in the situation. Instead, they focused on another aspect mentioned in the situation. Thereby they addressed an additional, individual facet not relevant to the mathematical solution. The transcript starts when Fynn and Svenja have received their task.

223	Svenja	*[5s]* Are both (situations) for Sina?
224	Fynn	What?

225 Svenja Does it *[looks at the RipOff situation]* say something about Sina, because here *[points to the questions]* it says Sina?

226 Fynn No.

227 Svenja Then these *[points to the questions "After how many months does Sina pay 84.99 Euro?" and "What is Sina going to pay in one month if she has bought 5 films?"]* might theoretically belong to this *[points to the RipOff situation]* because here the text – um – also mentions a Sina. A girl. And each of these two *[points to the questions "After how many months does Sina pay 84.99 Euro?" and "What is Sina going to pay in one month if she has bought 5 films?"]* mention Sina. That's why I think that they belong together.

Svenja focused on the name ||Sina|| that appeared both in the RipOff situation and in the questions. However, she seemed to notice the name in only two of the four questions. Hence, Svenja only referred to these two questions when assigning them because of the mentioned ||Sina|| in line 227. In this way, she did not use any facets of the core of the function concept to interpret the situation. Instead, she addressed an additional facet that is not relevant to the mathematical quantities or the function to explain her assignment of questions to the situations, the name ||Sina||. Hence, she did not identify the relevant verbal representation that describes the ||functional relationship|| and thereby the ||involved quantities|| in the situation.

4.3 Brief Summary of the Case of Mike and Nils

Mike and Nils (both 15 years old) successfully worked on the task and assigned the questions to the situations adequately.

92 Nils *[4s]* Mm. Everything concerning months belongs to this *[points to the RipOff situation]* I think. Then this *[points to the question "What is Sina going to pay in one month if she has bought 5 films?"]* belongs to this *[puts it to the FlatWatch2.0 situation]*. Yes, here *[looks at the RipOff situation]* is depending on the months, isn't it?

93 Mike Yes.

94 Nils Good and here *[looks at the FlatWarch2.0 situation]* it's depending on the films bought. What does she spend in one month when she has bought four films? (…) How many films does Sina – must pay 59 Euro. Yes, that fits.

Mike and Nils focused on several facets. Nils' first criterion for assigning the questions to the situations is the ||involved quantity|| "months". In line 92 he referred to the quantity "months" in both the question ("everything concerning months") and in the situation ("depending on the months"). Similarly, he explained in line 94 the fit of the other two questions to the FlatWatch2.0 situation. He identified the ||involved quantity|| "films bought" in the situation ("depending on the bought films") and in the questions. He identified the verbal representation containing the linguistic means "depending on" and interpreted the second part of the sentence with regard to the ||involved quantity|| that has the role of the ||independent variable||. He then decided whether the question fits the situation or not. This was sufficient and adequate to solve the task. Only the second part of the task (answering the questions) could show whether he would notice the ||direction of dependency|| adequately (Due to a lack of space, this analysis will not be shown here).

The case of Mike and Nils shows a successful approach to connecting verbal and symbolic representations in the situation, where the boys considered the ||independent variable||. In this case, this was sufficient to find a solution to the task. However, they did not check that the second ||involved quantity||, here in the role of the ||dependent variable||, was equivalent in both the situation and the question. This could possibly lead to wrong assignments in other situations.

4.4 Brief Summary of the Case of Altin and Jona

Altin and Jona (both 15 years old) assigned the question "After how many months does Sina pay 84.99€?" to the RipOff situation and the question "How many films did Sina buy, when she has to pay 59€?" to the FlatWatch2.0 situation. Jona explained this in the following way:

112	Jona	Yes, because here *[points to the RipOff situation]* it doesn't say that you can buy films. And that's why this belongs here *[points to the FlatWatch2.0 situation]*.

Based on the perception that the RipOff situation does not mention the number of films bought as one of the ||involved quantities||, Jona argued that the question of how many films Sina bought as ||unknown quantity|| does not fit to the RipOff situation and hence has to belong to the other situation FlatWatch2.0 (#112).

When the teacher asked Jona to assign the other two questions, Jona answered that the questions ("What is Sina going to pay in one month if she has bought 5 films?" and "What is the total price after 6 months?") fit both situations.

114	Jona	I think, they both fit.
115	Teacher	Why?
116	Jona	Because it is not specific – well – focused on anything that appears only in one of these *[points to the two situations]*. But it's simply put more generally.

Jona explained his assignment by arguing that the questions are not "specific" (#116) but "more general". Due to the context one can assume that Jona may have read the questions in a shortened way ("How many films?", "After how many months?"). In both questions, he adequately assigned before, the ||independent variable|| is the ||unknown quantity||. In the two questions he identified as "more general", the ||dependent variable|| is the ||unknown quantity||. In addition, these questions mention the ||independent variable||, but only towards the end of the questions. If he did read the questions in a shortened way, he might not have noticed the ||independent variable|| in the questions. This could be the reason why he described the questions as "more general" if the ||independent variable|| is his only criterion for assigning questions.

Altin objected in line 117 that the question "What is Sina going to pay in one month if she has bought 5 films?" contains the same ||quantity I||, namely the number of films bought.

117	Altin	But here it says *[points to the question "What does Sina pay in one month when she bought five films?"]* after five films. So when she –
118	Jona	Bought – Oh. Okay. Then this *[points to the question "What is Sina going to pay in one month if she has bought 5 films?"]* belongs to this *[points to the RipOff situation].*
119	Teacher	Why are they not the same anymore?
120	Jona	No – um. Yes here *[points to the question "What is Sina going to pay in one month if she has bought 5 films?"]* it says films bought and – here *[points to the RipOff situation]* you cannot buy any films and here *[points to the FlatWatch2.0 situation]* you can.

Jona's reaction "Bought – Oh" (#118) indicates that he did not notice this part of the question before. Furthermore, he revised his assessment with the same explanation as he did before when assigning the first two questions, namely by addressing the ||independent variable||. This indicates that he missed this part of the question by focusing on the first part.

In this way, Jona partly identified the relevant linguistic means in the situations and thereby the ||independent variable|| in the situations and as one of the ||involved quantities|| in two of the four questions.

4.5 Summary

These summaries and excerpts from four case studies show different approaches to assigning questions to situations. Some students did not identify the mathematically relevant verbal representation, as Svenja illustrates by focusing on the name "Sina". As the example of Jona shows, other students had problems interpreting the situations or questions regarding the ||involved quantities||.

Generally, the explanations of students' assignments vary with regard to the facets addressed. Some learners address facets of the core of the function concept flexibly and adequately. Other, less successful, students focused on only one of these facets; others did not focus on any of the facets. Nevertheless, most learners explained their assignment by referring to one of the ||involved quantities||, namely the one with the role as ||independent variable||.

These excerpts suggest that learners' processes of connecting representations work with regard to the ‖involved quantities‖. Potentially, learners regard these ‖involved quantities‖ in their role as ‖varying quantity I‖ and ‖varying quantity II‖ as well. This can be seen when learners decide whether one can answer certain questions with the given functional equation. In order to answer the questions, it would be necessary to decide, which one is the ‖independent variable‖ and which one is the ‖dependent variable‖, which requires addressing facets on a more compacted level. This allows one to plug in a value for the given variable and to calculate the value of the ‖unknown quantity‖. Whether they are also able to address the ‖direction of dependency‖ is shown in the second part of the task, analysed elsewhere (cf. Zindel, in preparation).

The examples presented illustrate how solving function word problems is a complex task that requires several steps. In addition, learners' descriptions became more precise when they differentiated the situations and questions in order to connect them explicitly. When designing instruction, it might be helpful to separate these steps and make them explicit. In order to assign questions to situations learners need to focus on the ‖involved quantities‖ in both the questions and the situation. These examples show that even this first step can be challenging. However, the task of assigning questions to situations has the potential to raise students' awareness of how to interpret the text in word problem and focus on the mathematically relevant aspects of the verbal representations. Making this first step explicit might lead learners to focus more on the ‖involved quantities‖, which is an important basis for solving word problems.

5 Conclusion

Solving function word problems requires identifying the relevant verbal representation in the situation and its interpretation with regard to other representations and the question posed. Therefore, conceptual understanding is necessary to decide what information in the text might be relevant in the situation. In order to solve a function word problem, it is necessary to identify the ‖given quantity‖ and the ‖unknown quantity‖ in the question and the ‖independent variable‖ and the ‖dependent variable‖ in the situation. These identified pieces of information need to be connected in order to be able to calculate the ‖unknown quantity‖. A first step in this process is identifying the ‖involved quantities‖ before interpreting their roles. Learners instead show different approaches as illustrated in Sect. 4. For some learners it might be helpful to identify and separate the different steps in this complex task.

In order to gain the information about the ‖independent variable‖ and the ‖dependent variable‖ it is often necessary to connect two or more representations of the functional relationship in the situation. However, the connections between the verbal and the symbolic representations, crucial for dealing with word problems, is not obvious to learners, and may be associated with topic-specific obstacles. This detailed analysis of students' learning processes showed which facets students addressed in order to assign questions to situations that contain both verbal and symbolic representations. Whereas some learners interpreted verbal representations by adequately addressing facets of the core of the functional relationships, other learners did not identify the relevant information in the verbal representation. Focusing on the facets of the core of the function concept has the potential for supporting learners as they deal with both the conceptual and language demands in word problems, and thereby deepen their understanding of the function concept (Prediger & Zindel, 2017; Zindel, in preparation).

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The Interplay of Language and Objects in the Process of Abstracting

Marei Fetzer and Kerstin Tiedemann

Abstract Abstracting is an important mathematical process. Students should think and talk about abstract mathematical entities, such as mathematical objects, relations, and processes. With regard to the process of abstracting, concrete didactic material and an abstract decontextualized language are often conceptualized as opposites. In this paper, we offer a theoretical framework that allows a new perspective on the interplay of language and objects in the process of abstracting. We combine three theoretical perspectives, namely the perspectives of empirical abstracting, of recontextualization and of objects as actors to analyse data collected in mathematics lessons in primary schools and, in addition, in tutorial sessions at the university.

1 Introduction

Open the door to any primary mathematics lesson and you will see how young students write on paper, look at books or work with manipulatives. You will hear them explaining their mathematical ideas, asking questions, and giving answers. So you will quickly realize that both objects and language are constitutive parts of mathematics learning at the primary level. Learning mathematics without using any language or without handling manipulatives does not seem to be possible at all. However, when the students get older and move to higher grades, the use of concrete objects decreases and activities connected to them become less important. They are replaced by an increasing emphasis on speaking and writing. Language becomes more and more important and for this reason has to be developed by the students. Whereas colloquial language is well accepted in early grades, students

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© Springer International Publishing AG, part of Springer Nature 2018
J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_8

need to develop more elaborated language in higher classes. From a theoretical perspective, this sketchy description of changes is often linked to the process of abstracting from the concrete (actions) to the abstract (thinking). Students start by working on a concrete level, but, at some point, they should think and talk about abstract mathematical ideas like numbers, two-dimensional figures or probabilities—without any reference to concrete objects.

As researchers in mathematics education who focus on primary school we ask: What roles do concrete language and objects play in the early processes of abstraction? How do they interact in children's learning processes and how can they benefit from each other? With regard to these questions, we present a theoretical framework in this chapter that allows a new and useful perspective on two aspects that are so typical for mathematics classes at primary school: language and objects.

2 Basic Assumptions on Mathematics Learning

2.1 About the Setting: Mathematics Learning as a Social Process

Together with many others we understand mathematical learning as a social process (Blumer, 1969; Cobb & Bauersfeld, 1995; Miller, 1986). It takes place in social situations and is always connected to a specific social context. In this regard, Miller (1986) stresses that substantial learning is only possible in interactional processes. Only if the individual learning process is a constitutive part of a process of social interaction, the individual may learn something that is substantially new to him (p. 5). Thus, mathematical learning is a matter of participation, and learning is a matter of meaning-making. According to Blumer (1986), subjective meaning-making of situations, symbols, and objects always takes place in interactions. In addition, meanings are developed, adapted, and changed with relation to the meanings which other participants of the interactional process create and share. These ideas are based on the approach of symbolic interactionism. Blumer (1986) gives an outline of symbolic interactionism, identifying three premises:

1. "Humans act toward things (including other individuals) on the basis of the meanings the things have for them" (Blumer, 1986, p. 2).
   Thus, humans interpret specific actions, things, and situations. Their subjective meaning of that action, thing or situation might differ from another person's interpretation. It is important to notice that Blumer's (1986) use of the term 'thing' differs fundamentally from the understanding of 'things' throughout the rest of this chapter. It is as broad and overarching as possible. Blumer defines: "Such things include everything that the human being may note in his [sic] world—physical objects […], other human beings […], institutions […], guiding ideals […], activities of others […] and such situations as an individual encounters in his [sic] daily life." (p. 2). In contrast, we will apply the

everyday-term 'thing' below with regard to actor-network-theory in a much closer sense.

2. "The meaning of things arises out of the social interactions one has with one's fellows [*sic*]" (Blumer, 1986, p. 2).
   Blumer's (1986) second premise refers to the source of meaning. Meaning is not inherent in the thing. Nor is it a psychical accretion like a sensation, memory, or feeling brought into play in connection with perceiving the thing. Instead, "symbolic interactionism sees meaning as arising in the process of interaction between people. The meaning of a thing for a person grows out of the ways in which other persons act toward the person with regard to the thing." (p. 4). Thus, the meaning of things is formed in the context of social interaction. It is seen as a social product.
   The meaning of a thing is derived from the interactional process. However, meaning is not an already established application to a thing. Meaning does not arise from the thing itself. In contrast, the use of meaning by the actor occurs through a process of interpretation. This leads to the third fundamental premise put forward by Blumer (1986).
3. "Meanings are handled in, and modified through, an interpretive process a person uses in dealing with the things he or she encounters" (p. 2).
   Thus, interpretation becomes a matter of handling meanings. Humans attribute meanings to objects, act according to these meanings, and then revise the meanings to guide their future actions.

For our studies on the interplay of language and objects, we essentially adopt two assumptions from this theoretical framework, namely:

- Individuals act towards humans and non-humans on the basis of the meanings that those things have for them (Blumer, 1986, p. 2ff). Thereby, a 'thing' can be anything that one can refer to, for example a student, the teacher, some beads, a number, or a non-human action.
- Such meanings arise out of processes of social interaction that one has with others, no matter whether those others are humans or objects.

2.2 About the Content: Mathematics Learning as Abstracting

On the surface, it might be surprising that many concrete objects are present in usual mathematics classes, as many mathematics teachers agree that mathematics is not about concrete objects, but about abstract objects. This means that mathematics is "a self-contained system separated from the physical and social world" (Mitchelmore & White, 2004, p. 329). Mathematical objects take their meanings only from the system within which they are defined. Thus, children cannot touch a number, a two-dimensional figure or the probability of an event, but they should learn about precisely these things in their mathematics classes. They should understand how those objects are defined in mathematics. And children can do this with the help of manipulatives because those manipulatives stand for mathematical (abstract) objects. The manipulatives refer to those abstractions. This means that manipulatives are concrete in nature, but they have the potential to represent abstract objects. In this regard, objects are important: They help to clarify the topic of a mathematical discourse (Sfard, 2008, p. 147). They help students and teachers talk about numbers, figures, and probabilities. In our opinion, this is a basic assumption on learning mathematics at the primary level: learning mathematics relies on concrete objects, but at the same time, it means to abstract from them.

In order to better understand this point, we refer to the idea of *empirical abstracting* from Skemp (1986). He also assumes that learning mathematics means to learn about abstract objects. To use his words, we cannot learn mathematics directly from our everyday environment, but "only indirectly from other mathematicians" (p. 30). Evidently, Skemp shares not only our assumption on the abstractness of mathematical objects, but also our assumption on learning mathematics as a social process. According to him, you do not learn mathematics alone, but in continuous exchange with others. And it is in this social process that students construct abstract mathematical concepts.

Skemp (p. 19) assumes that we always bear our past experience on the present situation. We make new experiences again and again, but in some experiences, we notice some invariant properties. That means that we recognize similarities ("in the everyday, not the mathematical, sense", p. 21) among our experiences. If we become aware of such a similarity, we can name it and, thereby, form a new concept. From then on, we are able to recognize new experiences as having something in common with an already formed class. We can classify this new experience right away although it is new. According to Skemp, this is a very important characteristic of mathematics learning: individuals recognize similarities—not in terms of appearances but in the underlying mathematical structure.

Mitchelmore and White (2007, p. 2ff.) refer to this concept of abstracting as empirical because it starts from the investigation of real world situations. They distinguish it from the concept of theoretical abstracting. In order to clarify the latter, they quote Davydov (1990, p. 25) and describe theoretical abstraction as a process of "a mental and systemic analysis of the relations and connections among objects" (Mitchelmore & White, 2007, p. 4). This means that theoretical abstracting starts already from abstract objects: from relations. For that reason, the distinction between empirical and theoretical abstraction helps us to see more clearly what we would like to investigate in mathematics classes. We are interested in processes that are typical for primary classes: students and teachers use objects to clarify the topic of their mathematical discourse and, therefore, have to abstract empirically. They have to reach agreement on the question of which similarities they 'see' in their experiences with objects. For that purpose, students and teachers need language. They have to talk about their interpretations of those objects.

3 Background Theories on Language and Objects

3.1 Functions of Language (Bruner, 1974)

The need for talking about different interpretations of objects has direct influence on our concept of mathematics learning. We understand mathematics learning as a social process that aims, at least in part, at empirical abstraction in Skemp's (1986) sense. That is why students and teachers have to use language. It is their medium for exchange about abstract mathematical objects. Furthermore, language is not only a medium for exchange, but it also serves other functions (Bruner 1974, p. 106). For the context of our research project, we take up a differentiation between two functions that we find in Bruner's (1974) considerations for a theory of instruction.

The first function has already been mentioned: language is a medium of exchange (p. 6). It allows students (and teachers) to share their mathematical ideas with each other. Students can express by linguistic means how they interpret the particular mathematical topic and, then, get feedback on their thinking: does it already meet the mathematical conventions or are there any possible or even necessary improvements? Thus, language is a representation that allows students and teachers to negotiate an agreement on mathematical topics and on the appropriateness of related interpretations. This is probably why Bruner (1986, p. 21) states that "the heart of the educational process consists of providing aids and dialogues". Consequently, mathematics classes can be understood as a special form of dialogue. In this dialogue, language is not the only representation, but a very important one. Bruner (1986, p. 105) refers to language as a "powerful instrument".

The power of this instrument is not only to facilitate a discussion about mathematical ideas, but to help children improve their mathematical thinking at the same time. This is the second function of language. Bruner (1986, p. 103) investigates "how the use of language affects the use of mind". He explains that "the speaker or writer rides ahead of rather than behind the edge of his utterance." The one who uses language has to organize his or her thinking in a way that allows him or her to produce a clear and comprehensible utterance. The person has to structure his or her thinking and has to anticipate what needs to be said. In this sense, the use of language challenges students to rethink their ideas and thus helps them develop their mathematical thinking further. Thus, students use language in order to share their ideas with others (first function) and this use of language, in turn, affects their ideas (second function). In this regard, the two functions of language focus at the link between communicating and thinking and are related to each other.

With his description of the functions of language, Bruner does not focus on any specific content. This is why we connect his concept with one that focuses on the content level.

3.2 Language as a Means of Recontextualization (Aukerman, 2007)

Especially with regard to second-language learners, Cummins (2000) has argued that children have to develop different types of language skills during their lives. First, they need *basic interpersonal communication skills* (BICS), which are adequate for a 'playground talk'—i.e., for informal face-to-face interactions about everyday topics. But later on, they need *cognitive academic language proficiency* (CALP) in order to cope with cognitively demanding topics in the institutional setting of school. Thus, a change of language use becomes necessary due to a change of learning settings. While children learn from direct experience outside the school, they start to learn first of all from texts and teachers in school. In order to be able to learn from those now indirect experiences, students have to *decontextualize* their language use and, for that reason, have to develop more and more complex linguistic structures.

To complement this distinction between BICS and CALP, Aukerman (2007, p. 630) points out that it is misleading to talk about a decontextualized language because no "text, and no spoken word, ever exists without a context." Thus, in contrast to Cummins (2000), Aukerman does not focus on the setting of language production or reception, but on the content-related context: She asks which topics individuals refer to by using language. From the perspective of an individual, contexts may differ in many ways; they may be concrete or abstract, well-known or unfamiliar, or even completely new, but Aukerman stresses that we always talk and listen to others with regard to a specific context. To use her words, we *recontextualize* every utterance. No matter whether we are the ones who speak or the ones who listen, we relate every utterance to a context that we regard as adequate at that very moment. Thus, when students are expected to talk about their mathematical thinking, they have to re-contextualize their everyday language and have to transform it to more abstract contexts. Seen from that perspective, the question is no longer whether a student is able to decontextualize his or her language, but the question is now whether individuals in the classroom succeed in finding a common context. Do their re-contextualizations fit sufficiently together?

3.3 Objects as Actors (Latour, 2005)

Introducing Actor-Network-Theory (ANT), the sociologist Bruno Latour literally demands to "reassemble the social" (2005). In his book, he suggests a perspective that accepts concrete objects and things as actors. He considers objects as well as humans as full participants in the course of action. This conceptualization of objects and their role in the emergence of social reality is not in line with the traditional view of sociology, and although it may be unfamiliar to many in the field of mathematical classroom research, it is becoming increasingly known (for example, see deFreitas & Sinclair, 2014). However, in the context of the focus of our research, Latour's approach fits perfectly: We understand mathematical learning as a social process. Processes of empirical abstraction are situated in a social context. Who are the actors in this social environment? Taking the idea of social learning seriously, it seems appropriate to re-think the question of participation in the emergence of social reality, follow Latour, and take objects and things into account socially.

Objects as actors? This is in short one of the key-assumptions Latour suggests. What does he mean? In order to understand his idea of reassembling the social it is helpful to have a closer look at Latour's conceptualization of "the social". Latour goes beyond the traditional understanding of "the social" and actually redefines the notion (2005). He takes a closer look at whom and what assembles under "the umbrella of society" (p. 2). To him, sociology is not only "the science of living together" (p. 2), and the social is not only about the assemblage of people. Instead, according to him, looking on the social means "the tracing of associations" (p. 5). In his view, the social refers to any kind of networking: humans and humans, as well as humans and things. Heterogeneous elements which are not necessarily social themselves associate in different ways. According to Latour, all these different associations create social reality.

In his Actor-Network-Theory (ANT), Latour (2005) extends the list of potential *actors* in the course of action fundamentally and accepts all sorts of actors: "Any thing that does modify a state of affairs by making a difference is an actor" (p. 71). Consequently, objects participate in the emergence of social reality, too. In this sense, Latour asks for a broader understanding of *agency*. "Objects too have agency" (p. 63). They are associable with one another, but only momentarily. To use Latour's words, they "assemble" (p. 12) as actor entities in one moment and combine in new associations in the next one. Accepting objects as participants in the course of action, Latour gives up the idea of stable and pre-defined associations and actor-entities. He reassembles the social.

Following Latour (2005) in accepting objects as actors in the process of interaction and in the emergence of social reality, we have to reassemble the social. With regard to the social in the mathematics classroom, we now have to pay attention to the fact that—seen through Latour's sociological glasses—objects can participate in the social process of negotiating a common context and can, in this way, influence processes of empirical abstraction. Following Latour, objects are much more than just tools which students can use with a specific intention, but they can assemble as actor entities and make a contribution to the ongoing social interaction. "Any thing" (p. 71), a human or non-human actor, might become associated with other actors in the course of action, but only momentarily. The association might be dissolved the next minute. In fact, this is true for all sorts of objects: paper and pencil, as well as manipulatives or even the bottle of water on the table. Should we as researchers in mathematics education not focus on a certain kind of object, on didactical material? From a theoretical as well as from a methodological point of view, we clearly deny that restriction. Just imagine that the bottle was open, and would drop. Not only the table, but also the paper would get wet, the pencil might fall on the floor.

This would surely influence any process of social interaction. However, in that very moment these actors, no matter who and what they are, contribute to the ongoing process of social interaction and influence the process of mathematical learning.

Looking through Latour's sociological glasses, we can see clearly that concrete objects do play a role in the emergence of social reality. However, it should be noticed that objects' mode of action is different from the way human participants contribute to the social. But, how *do* they contribute? Fetzer (2013) describes different forms of participation that object-actors realize in interaction processes. Our current research on the interplay of language and objects goes one-step further. Now, we try to grasp the contributions of objects on the content level.

Mathematics education has to deal with all sorts of (material) objects, didactical tools, artefacts and manipulatives, diagrams, and signs. All those objects leave their traces in the emergence of mathematical learning processes and take part in the course of action. Mathematical learning appears to be closely connected to objects and non-human things. Even if Latour himself does not suggest any methods of empirical analysis, Latour's approach provides a fruitful background theory for empirical research in mathematics education (Fetzer 2013). Accepting objects as participants in the course of action and following the idea of objects having agency helps us to get a better understanding of mathematical learning processes.

4 Methods

Analysing classroom interactions with reference to the framework of symbolic interactionism is a matter of interpretation. It is an interpretative effort to reconstruct processes of meaning making. How is meaning formed and negotiated in the process of interaction? How do actors collectively create mathematical meaning? In order to investigate the process of empirical abstraction, we apply the analysis of interaction (Cobb & Bauersfeld, 1995). This method refers to the interactional theory of learning and is based on ethnomethodological conversation analysis (Sacks, 1996). A working group directed by Bauersfeld originally devised this method. In contrast to conversation analysis, it focuses on the thematic development of a given face-to-face interaction rather than on its structural development. It reveals how the sequential organization of interaction is constituted. Every single action is interpreted extensively in the sequence of emergence. The analyst tries to generate as many alternative interpretations as possible. Thus, he or she opens up the range of potential ways of understanding and construing the action. In order to get hold of the process of interacting, actions are considered to be related to each other. They are interpreted as turns to previous actions (Sacks, 1996). Analysing turn by turn, the process of meaning making can be reconstructed. Traditionally, interaction analysis captures human actors as participants of an interactional process and investigates their actions. In the context of our research, we follow Latour (2005) and regard objects as actors, too (Fetzer, 2009, 2013).

As explained above, we focus on mathematics learning at the primary level. In that context, it is not that important for us to observe specific lessons, but rather to observe a wide range of occurring mathematical learning situations. For that purpose, we filmed several mathematics lessons in three German primary schools (cf., Sect. 5.1 “Result I”) and, in addition, tutorial sessions at the university (see Sect. 5.2 “Result II”). On the basis of those videos, we identified the scenes in which both humans and objects participate. We chose those scenes in which human actors associate themselves with objects by talking explicitly about them or pointing at them. Such scenes were transcribed and became our objects of analysis.

5 Connecting Theories

In the following, we present a networking of theories to outline our perspective on the interplay of language and objects in the process of abstracting. Afterwards we give two examples to illustrate the insights that this perspective allows.

According to Skemp (1986), you will have to learn from and with others if you want to learn about abstract mathematical objects. Thus, they are not directly accessible, but they are developed as a product of similarity recognition and of mathematical definition. Mathematicians have themselves developed, negotiated, and revised the meanings of mathematical objects through their social and historical interactions within the profession. This mathematical system is essentially separated from the physical and social world (Mitchelmore & White, 2004, p. 329). As a consequence, you need words and other signs in order to get access to it. So if children want to learn about those mathematical conventions they will have to participate in the social process called mathematics instruction. There, students and teachers use language in order to negotiate their interpretations of mathematical objects (Blumer, 1974, 1986). They have to come to an agreement about the context of their mathematical discourse (Aukerman, 2007): what are they talking about and do their re-contextualizations fit sufficiently together? In that very challenging interaction, objects offer their help in the form of different possibilities of assemblage. Students and teachers are invited to choose the offer that fits best to their similarity recognition.

5.1 *Result I: The Interplay of Language and Objects*

A short example from a second grade class illustrates how different forms of interaction between students and objects take effect on the interplay of language and objects in a mathematical discussion (Fetzer & Tiedemann, 2016). The students and the teacher focus on the hundred board in front of the classroom and talk about the question of what “the diagonal” might be (Fig. 1). In this particular case,

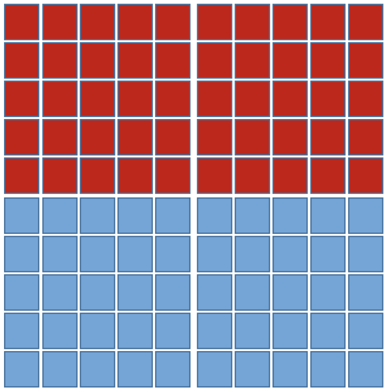

Fig. 1 Hundred board covered with pieces of paper (red-coloured in the upper half, blue-coloured in the lower half)

the hundred board does not show the numbers from 1 to 100, but 50 red squares in the upper half and 50 blue squares in the lower half. Thus, all numerals are covered with pieces of paper.

Following Latour (2005), we can regard the hundred board as an actor and recognize different offers that it makes:

1. "The diagonal runs like this."
2. "The diagonal runs from 10 to 91."
3. "The diagonal runs from one corner of the hundred board to the opposite one."
4. "The diagonal runs from one corner of a square to the opposite corner."
5. "The diagonal connects two nonadjacent vertices of a polygon."

It is possible to assign these offers to different levels of abstraction. While some of them are more concrete, others are abstract in the mathematical sense. But for the social situation of the mathematics lesson, it is exclusively important which of these offers is accepted by the students or the teacher. Which 'definition' of a diagonal will become an object of discourse? For that purpose, students have to talk with 'the help of' the object. What we could find in our data is that students can interact with objects in two senses.

On the one hand, students can *act in association with objects* (Fetzer & Tiedemann, 2016). When children pick up a pair of scissors or fold a piece of paper in order to prove the symmetry of a figure, child and object assemble in their action. In these cases, objects take over part of the turn. Action is no longer bound to one single actor, but to actor entities. Thus, actions are experienced as combined actions. Referring to the example of the hundred board, one child might touch the board and show with the finger: "The diagonal runs like this." In that situation, the student acts with the help of the object hundred board. If objects and students assemble in their actions and humans have to take over only part of the turn, children are relieved on the language level. They can use structurally more simple language without losing comprehensibility. They refer to the object and the object 'tells' the rest. Thus, the assembled action appears to be complete so that other participants can easily reconstruct the meaning of the utterance. Again, this is interesting from a didactical point of view. If objects participate in the mathematical discourse, then students can share their mathematical ideas, even if their mathematical language is (possibly) still in an early stage of development. But this *acting in association* includes a potential for the future development of mathematical language: as long as students and objects perform combined actions, the offers which students accept remain specific. But, if the children start to look for similarities in different experiences, they will have to refine their language and take over more and more turns on their own.

On the other hand, students can talk with objects in terms of *interacting* (Fetzer & Tiedemann, 2016). That means that human actors and objects interact in their turns. In such a situation, objects take over whole turns and students react to them. A child might, for example, react to one offer of the hundred board and utter: "The diagonal runs from one corner to the opposite one." If human and non-human participants interact in this way, then objects challenge students' language. We could observe that, in these situations, spoken language tends to be similar to written language because aspects of the situational setting have to be made explicit by the child. As a consequence, the structure of the language is more complex. This aspect is interesting from a didactical point of view: objects can challenge students on the language level. They can serve as 'supporters' in the development of mathematical thinking and in the development of more precise use of mathematical language.

5.2 Result II: The Interplay of Language and Objects in the Process of Abstracting

A second example illustrates how the interplay of language and objects can be productive for the process of abstracting (Skemp, 1986). The scene is taken from a tutorial session at the university where Hanna and her tutor Britta work together. Hanna is 9 years old and takes part in a project for students with mathematical learning difficulties. She comes to university every week for about an hour and works with Britta who is specially trained for that purpose. In their lessons, Hanna and Britta work with two different types of manipulatives: a bead frame and Dienes blocks. In one of their later sessions, they start to compare the two types of objects. Hanna is asked to look for a similarity between the bead frame and the Dienes blocks. In the following example, Hanna and Britta talk about the representation of the number 42. Britta has the Dienes blocks in front of her and asks Hanna how to represent the 42. (The conversation was translated from German by the authors.)

Britta	And how do I put the 42?
Hanna	Uhm, 42, so you put 4 bars of tens and 2 single cubes.
Britta	[*nodding*] Fine. [*puts 4 bars of tens and 2 single cubes on the right side of the bars*] And how would I put a 42 on the bead frame? Can you tell me that, too?

Hanna Yes, just the same. Because 42 is 42. But you take rows and not bars, okay? That doesn't matter. And then, and then … 42 on the bead frame… And then, 2 pearls. Then, you have 42.

Britta Yes, great.

In the tutorial sessions, Hanna has refined her mathematical thinking and her mathematical language. Concerning the bead frame, she distinguishes tens and ones and talks about them as 'rows of tens' and single 'beads'. Similarly, with the Dienes blocks, Hanna distinguishes tens and ones and talks about them as 'bars of tens' and single 'cubes'. In the scene at hand, Hanna interacts with both objects at the same time. While the Dienes blocks lie in front of her, the bead frame is not even present. Nevertheless, Hanna obviously recognizes a similarity between the two objects. Both the bead frame and the Dienes blocks offer her the possibility to clarify what a 'ten' actually is. In the case of the bead frame, a ten is a row of ones. In the case of the Dienes blocks, a ten is a bar of ones. But in that scene, rows and bars have become exchangeable for Hanna. She explains that they both represent a ten: "just the same". In order to describe this similarity Hanna relies exclusively on language. She uses words for her empirical abstraction and takes over the whole turn on her own. Thus, the interplay of language and objects may be especially productive for the process of abstracting, if children start to interact with at least two different objects at the same time, no matter whether the objects are physically present or only present in one's imagination.

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Intersecting Language Repertoires When 4-Year-Olds Count

David Wagner and Annica Andersson

Abstract In this chapter, we consider an interaction among a researcher and four 4-year-old boys who were asked to count beans. By recognizing multiple discourses at play, we problematize the identification of this (and other) interaction as a mathematical (or counting) situation. We identify aspects of the children's language repertoires and consider how they index various discourses and authority structures. We ask how these discourses intersect in the interaction, and then identify potential implications for education practice and research.

1 Context

In this chapter, we analyse an interaction among a researcher (David) and 4-year-old children, to illustrate the way multiple discourses are at play, and to problematize the identification of interactions as mathematical (or counting) situations. The classroom in which this interaction took place was established decades earlier to model exemplary early childhood education while giving opportunities to emerging educators and researchers to understand early childhood education better. With his office around the corner from the classroom, David had visited it earlier to talk about a stick that he likes, to talk with the teachers, and in costume on a costume dress up day. The children often greeted him in his office when they walked by. In other words, the people in the interaction were somewhat familiar with each other.

Another important aspect of the context of the interaction was the research study that motivated David. He brought with him a video camera and voice recorder,

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© Springer International Publishing AG, part of Springer Nature 2018
J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_9

operated by himself, at a table to which children were invited by the teachers to choose to visit. These recorded interactions were part of a larger study with the expressed aim of identifying the mathematical language repertoires of children. The goal was to better understand the way children experience mathematics across age and linguistic background spectrums. How this chapter problematizes the common-sense idea of a mathematical situation is not only a critique of others, since even the original research project was underpinned with this same common sense idea.

The children brought with them their own motivations and interests, which are only available to us through their language and other actions. However, we recognize that the presence of four researchers (David, Annica, and two assistants), three with video cameras on this day, marked this situation as different than David's other interactions with the children. These factors undoubtedly influenced the children's interpretations of the situation and thus also their motives and interests.

David had a container full of dry, uncooked kidney beans (relatively large, solid dark red), pinto beans (medium-sized, light beige with dark spots), and navy beans (relatively small, solid white). We (Annica and David) had chosen a variety of beans to prompt the possibility of children distinguishing among the different kinds of beans. We also had broken many of the beans into pieces to provoke the children to decide what counts as a bean. The variety of beans and presence of partial beans were intended to highlight aspects of the politics of counting; the decision of what to count is more apparent when the items counted include a variety of items and "abnormal" items.

When children came to the table, David dumped some beans into a pile and asked the children if they could count them or if they wanted to count them. We were both in the classroom, each hosting a separate table, but we focus on the interactions at one table for this paper—the table where David was the host.

2 An Apparently Mathematical Situation

David was the primary author of a funding proposal, a large-scale research project that mainly focused on identifying specificities of students' language repertoires, in contexts of mathematical investigation. However, this chapter focuses on problematizing that research plan's fundamental question. We can say with confidence that the interaction described in this paper was a situation, but was it a mathematical situation? We sought to identify connections between the children's ways of talking about counting and what their expressions may mean. However, we also identified a number of other discourses intersecting with the mathematical discourse.

It is easier to understand that the situation may be seen as another kind of situation, when we consider the perspectives of the children. Was it a mathematical situation for them? We will argue that each child would have seen the situation differently, and that the way they saw the situation changed from time to time throughout the interaction. Even for David it was more than a mathematical situation—for example, it was a research situation, and it was also an opportunity to enjoy playing with and getting to know some children whom he already knew to some extent, some of whom had parents he knew better.

In this paper, we will use episodes from our data as examples to show how these other discourses play into the mathematical conversations and thus connect with the language repertoires the children brought to this apparently mathematical situation. It is important for us to be aware of what we mean by mathematics, the location of agency in mathematics, and how we think about interaction among these agents and the discourses within which they operate.

2.1 Mathematical Situations from the Perspective of Positioning Theory

Positioning theory supports us to understand the different ways students' communication acts connect to a variety of discourses, including mathematics. A powerful aspect of the positioning theory developed by Harré, van Langenhove and their colleagues is its radical focus on the immanent and its associated rejection of the transcendent. In other words, it considers real only that which is present in the interaction and rejects the power of exterior forces. We identify this theory as powerful because it enables a focus on the relationships that are present and allows us to ignore the exterior, relatively intransigent forces outside the interaction. This view helps us identify the possibilities for changing available discourses. Davies and Harré (1999) used Saussure's distinction between discourse practice and the discursive systems in which practices are situated to illustrate positioning theory's radical focus on the immanent: "La langue is an intellectualizing myth—only la parole is psychologically and socially real" (p. 32).

However, in an analysis of the way positioning theory was taken up in mathematics education research, Wagner and Herbel-Eisenmann (2009) noted that the norm in mathematics education research was to identify students' positioning in relation to mathematics. Wagner and Herbel-Eisenmann reconciled this approach with positioning theory's rejection of exterior forces by saying that these exterior forces, such as the discipline of mathematics, may be myths, but they can be taken as real in classrooms or in other interactions because teachers and others may be viewed as representatives of these exterior forces. In this way, students may position themselves in relationship to mathematics, even though mathematics has no physical presence in the classroom. The teacher or a textbook may be seen as a medium of mathematics. However, allowing for the positioning of people in relation to mathematics undermines some of the power of positioning theory, as noted above. The same tension exists for other ideas that transcend the local interaction as we elaborate below (e.g., gender, school practices, and technological practices).

The classical triad developed by the progenitors of positioning theory (e.g., van Langenhove & Harré, 1999, p. 18), connects positioning with storyline and speech acts, placing them on a triangle together. Storylines draw on shared narratives that pre-exist an interaction. This triad was reconfigured by Herbel-Eisenmann, Wagner, Johnson, Suh, and Figueras (2015, p. 194), as shown in Fig. 1. They layered storylines to emphasize positioning theory's claim that multiple storylines may co-exist in an interaction. And they used arrows to highlight the dynamic interaction between a communication act and the exterior storyline—a communication act initiates, maintains, and negotiates positioning within a storyline, and this positioning formats communication acts. For us, this recursive relationship is reminiscent of Foucault's (1982) description of discourses—"practices that systematically form the objects of which they speak" (p. 52). Thus we generally use the term discourse instead of storyline.

Herbel-Eisenmann et al. (2015), who also emphasize this recursive relationship, elaborated the way these discourses interact with multiple discourses at once (see Fig. 1). We claim that the non-mathematical discourses are not impediments to mathematics any more than mathematics is an impediment to those other discourses. Rather, the discourses support each other (Andersson & Wagner, 2016). In short there are many discourses enacted in any classroom context. We claim that one cannot understand students' communication about mathematical processes without understanding that these acts are also part of their repertoires for the other discourses in play.

In the context we describe in this paper there was some counting going on, and thus there was a mathematical aspect to the interaction. We also consider other discourses at play, including ones relating to gender, ability/inability/disability, food, fantasy, canonical children's stories, techno-gadgetry, friendship, humour/clowning, and teacher/student relationships. These are the discourses we have

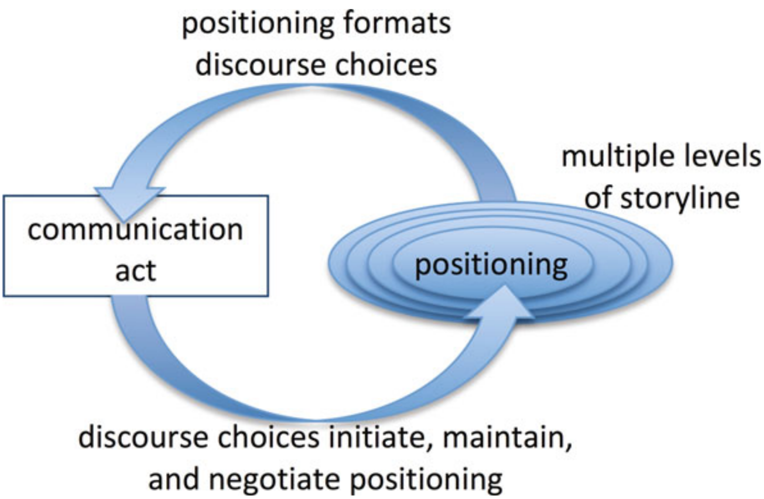

Fig. 1 Multiple discourses recursively at play in a situation (from Herbel-Eisenmann et al., 2015, p. 194)

identified in the interaction, but there are others we may notice later, and again others the children may have engaged that are unknown or unavailable to us. Thus, when we consider any *communication act* (the term chosen by Herbel-Eisenmann et al. (2015) to widen language acts to include other forms of communication), it is problematic to claim that it represents only the speaker's mathematical language repertoire. It may well do so, but it may also represent other discourses important to the speaker that they bring into a situation. The word *representation* is also problematic because, as noted by Herbel-Eisenmann et al. (2015) and Foucault (1982), communication acts do not only represent various discourses at play, they also initiate, maintain and shape those discourses.

3 Analytic Approach

For our analysis, we draw on an authority structure framework as described by Wagner and Herbel-Eisenmann (2014), which draws from their large-scale quantitative analysis of communication in mathematics classrooms (Herbel-Eisenmann & Wagner, 2010). This framework distinguishes among expressions of *personal latitude, personal authority, discourse as authority,* and *discursive inevitability.* Table 1 summarizes this framework and how to operationalize it in the analysis of communication.

With expressions of *personal latitude* it is recognized that people are making choices. We look for "evidence that people are aware they or others are making choices" (Wagner & Herbel-Eisenmann, 2014, p. 875). The linguistic indicators for personal latitude include open questions (Martin & White, 2005), inclusive imperatives (Rotman, 1988), and indicators of someone changing their mind—for example, *I was going to, could have*. Expressions of *personal authority* are those utterances in which there are no apparent reasons for people's actions except that they follow the wishes of one of the people in the interaction. In transcripts we look for "evidence that someone is following the wishes of another for no explicitly given reason" (p. 875). Linguistic clues for personal authority include the presence of *I* and *you* in the same sentence, exclusive imperatives (Rotman, 1988), closed questions (Martin & White, 2005), and choral responses. We find expressions of *discourse as authority* where there is explicit recognition of a force outside the interaction compelling certain actions in the interaction. We look for "evidence that certain actions must be done where no person/people are identified as demanding this" (p. 875). The strongest linguistic clue is the presence of modal verbs that suggest necessity—e.g., *have to, need to, must*. Lastly, we search for expressions of *discursive inevitability* in which there is a sense of only one possible direction for action but no explicit recognition of what compels this action. We find "evidence that people speak as though they know what will happen without giving reasons why they know" (p. 875). The modal verb *going to* is a strong indicator of this structure.

Table 1 Analytical guide for authority structures from Wagner and Herbel-Eisenmann (2014)

Authority structure	Linguistic clues	General indicators of the structure (that may not involve the particular linguistic clues previously identified)
Personal latitude	• Open questions • Inclusive imperatives • Verbs that indicate a changed mind (e.g., *was going to*, *could have*) • Constructions that suggest alternative choices (e.g., *if you want*, *you might want to*)	Look for other evidence that people are aware they or others are making choices
Personal authority	• *I* and *you* in the same sentence • Exclusive imperatives • Closed questions • Choral response	Look for other evidence that someone is following the wishes of another for no explicitly given reason
Discourse as authority	• Modal verbs suggesting necessity (e.g., *have to*, *need to, must*)	Look for other evidence that certain actions must be done where no person/people are identified as demanding this
Discursive inevitability	• *Going to*	Look for other evidence that people speak as though they know what will happen without giving reasons why they know

Awareness of the positioning among the participants in an interaction and the many storylines they draw on to negotiate that interaction brings to centre stage the wishes of the people involved. Thus we attend to the location of personal agency in the interactions described here. The first two of the structures described above locate agency within the interaction and thus we think of them as micro-political. The other two structures locate agency outside the interaction and thus follow the politics of one of the many larger discourses at play in the interaction, and thus we think of them as macro-political. This distinction aligns with positioning theory's distinction between immanent and transcendent forces. Positioning theory recognizes only the immanent as a force (Davies & Harré, 1999) but people in the moment of interaction can mediate and thus bring transcendent forces into the moment as described by Wagner and Herbel-Eisenmann (2009) in their elaboration on the theory. We will use the term *index* to describe this recursive connection between a communication act and a discourse—for example, "The children's choice of who to talk with may be a communication act that *indexes* gender discourses" (from Sect. 4)—though we recognize that any possible verb, including *index*, would seem to foreground either the representational or constructive aspects of the relationship between the local communication acts (la parole) and the discourse (la langue). Our theoretical framing reminds us that the children's choice developed gender discourses for them and us in the example we quoted, and concurrently their experience of gender discourses informed their choices for action.

In our account of the children's display of their language repertoires, we work with them as instances within the conversation, and thus avoid presenting the situation in a singular narrative. Presenting a series of episodes connected together in a narrative often favours one discourse, chosen by the researcher(s) and thus cleansed of other discourses that probably were present in the interaction. As an invitation to others to consider alternative discourses at play, we identify some of the discourses we saw at play as evidenced in the children's communication acts. Sometimes the timing is important (which then requires a little contextual information and narrative to set up the situation). We connect these instances of conversation to the four authority structures identified above.

4 Language Repertoires in the Situation

Four boys came to David's table over the course of his time hosting the table. On this day, with Annica hosting one of the tables, generally the boys came to David and the girls to Annica though most of the girls in the class chose to have interactions with David at various other times in the year. The children's choice of who to talk with may have been a communication act that indexes gender discourses. Undoubtedly these discourses were present throughout the interaction though no other communication acts at David's table specifically invoked for us a connection to gender (except gender-exclusive pronouns). We reference gender discourses first because this makes it easier to write about the other discourses in English with its gender exclusive pronouns, not because we think gender was any more important than the other discourses in the context.

Next, we consider any counting words and strategies evident in the interactions, because that focus motivated the interaction from our perspective. (We remind ourselves at this point that the children were likely otherwise motivated.) However, this focus on counting immediately draws our attention to ability discourses. Three of the four boys counted beans (Fig. 2 shows a child in this classroom counting beans). However, this does not mean that Patrick[1] (the other boy) was unable to count. Perhaps he was not interested in counting. He certainly expressed interest in the technical gadgetry at the table, and knowledge of human interaction involving such gadgetry, which he demonstrated by picking up the voice recorder and modelling ways of holding it to mimic photography and telephone conversations. As Patrick indexed this techno-gadgetry discourse, he also became increasingly aware of his friends' pleasure at his humour, and thus seemed to increasingly play up his mimicry, and apparently indexed a clowning discourse. In classroom situations, we have noticed that a child's choice to engage or not in an invited conversation stream is often taken as an indication of the child's ability or proficiency in that particular topic. We suggest that a child's engagement or lack of engagement

[1]Names of participants are pseudonyms.

Fig. 2 A child counting beans

may well be a better indicator of the child's inclination or current interest than of ability.

We highlight Patrick's ability and interest in various discourses here because they confound stereotypical interpretations of children with Down's Syndrome (like Patrick), but also because ability discourses connect to an indicator of personal latitude, particularly in relation to inclination. David's and the other boys' expectations for counting from Patrick may have been relatively low because of disability discourses. However, David had seen Patrick count before, yet Patrick did not show any interest in counting on this day. The other boys displayed mixed messages about an interest in counting. When they came to the table (Colm first, followed by Gavin, followed by Reece, with Patrick coming and going), David asked each of them if they wanted to count. Our first transcript excerpt represents Colm's response to the invitation to count the beans. He first demonstrated his counting abstractly (not counting beans), and then he counted some beans that David had dumped into a pile on the table.

1	David:	We're going to count them. Do you want to get someone else to count with you? Or do you want to count by yourself?
2	Colm:	I want to count by myself.
3	David:	Okay.
4	Colm:	Well, I can count higher than ten.
5	David:	Can you, can you show me?
6	Colm:	One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty, twenty-one.
7	David:	[*Laughs*] That's twenty-one, right?
8	Colm:	Yeah, thirty, thirty-one, thirty-two, thirty-nine, thirty
		[…]
14	David:	Now I'm going to put some [*beans*] out here.
15	Colm:	Oooh! What are you going to do with them?
16	David:	Ah, what do you think I'm going to do?
17	Colm:	A pile.

18	David:	A pile. How many?
19	Colm:	Well, I can't even tell because there's too many!
20	David:	Is there too many?
21	Colm:	Yeah!
22	David:	What if you really wanted to know how many there are, how would you do it?
23	Colm:	Count.
24	David:	Okay, let's see. Start with these, Start with those.
25	Colm:	One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty, twenty-one, twenty-two, twenty-three, thirty, thirty-one, thirty-two, thirty-three, thirty!

Colm answered David's question about whether he wanted to count by himself or with someone else, saying, "I want to count by myself" (turn 2). This was not necessarily the same as saying he wanted to count. Indeed, when David dumped a pile on the table and said, "How many?" (turn 18), Colm said, "I can't even tell because there are too many!" (turn 19). The modal verb *can* appeared often in this interaction. For example, Colm said "I can count higher than ten" (turn 4) shortly before exclaiming that there were too many to count. On the basis of his exclamation that there were too many to count, we might then wonder whether he really *could* count higher than ten: Was he able? After saying he that he could, he recited the numbers going from one to nineteen, jumped back to twelve and then continued to twenty-one (turn 6). But that was reciting numbers without reference to objects. The modal verb *can* has drawn significant attention in this research project as it is common to language repertoires of children across the age spectrum and because the verb carries multiple possible meanings, some of which have mathematical significance (Wagner, Dicks, & Kristmanson, 2015). Some of these meanings will be considered in the analysis below.

When Colm counted actual beans later (turn 25), pointing to the beans he was counting, he did not make such mistakes, at least not as soon—he got as far as twenty-three before losing count. Later, he proudly demonstrated his ability to count the beans while sitting on his hands. Colm's pride indexed an assessment discourse, as he seemed to want to impress David. The counting while sitting on the hands indexed a body discourse. Within this conversation, it seemed to be assumed that restricting the involvement of one's hands made counting more difficult. Looking at the beans and counting them (turn 25), and later pointing with eyes while sitting on hands was different than simply reciting numbers as a list of arbitrary signifiers (turn 6). The connection between the numbers and the beans indicated the presence of a non-personal discourse that was unlike a discourse as authority and unlike discursive inevitability because this non-personal discourse was immanent. This connection reminds us of Pickering's (1995) and Latour's (2005) identification of agency in material (objects and events) in addition to humans and discourses. Other researchers in mathematics education have also pointed to the agency of objects, for example de Freitas and Sinclair (2014) and Fetzer and Tiedemann (2018, in this volume).

Gavin and Reece said similar things about their ability to count using the verb *can*, but added further utterances, expressions, and statements that indexed ability. For example, Reece responded to the invitation to count with “I definitely can.” Also, Colm possibly gave agency to the beans by saying, “it *can* detach” (again a possible *material agency*, in Pickering’s (1995) words). Colm was manipulating two partial beans, trying to fit them together to make one bean. He seemed to be thinking about whether to count the pair of broken bean pieces as one or two. This suggested the significance of the beans in the assemblage—using Latour’s (2005) word for collective agency among humans and materials. As we noted above, these broken beans may also be seen as the researchers’ communication act because we carefully constructed the bean pile to raise the question about what would count as a bean. Thus the form of the beans—broken and whole, and of mixed variety (see Fig. 2)—may be seen both as a material in the interaction and as a communication act.

In addition to indexing ability, the verb *can* may be interpreted to index inclination (Martin & Rose, 2005). When it comes to Colm’s inclinations, the many discourses that were at play in this context warrant attention. For example, after Colm took a break from counting activities to take pleasure in Patrick’s clowning he could have said, “I can count beans again.” Perhaps gender discourses were influential in him being at this particular table instead of, for example, Annica’s. Nevertheless, while gender and other discourses may have influenced the children’s inclinations, we could also identify desire as a discourse—children follow their own inclinations.

Other important modal verbs that commonly appear in mathematics classrooms and which are tell tale markers of *discourse as authority* are “have to”, and “need to” (and “must” which is less common). There were no instances of these modal verbs in the interaction (except David saying that to eat these beans “we have to soak them in water for a long time and then cook them”). The absence of these modal verbs highlighted the question that motivated the research project: What strategies do the children use instead to index the compulsion of an external discourse? The answer to this question is rather complicated. It is especially complicated by the ambiguity that goes with these strategies in terms of which external discourses they were indexing. In the following transcript we point to some of the possible external discourses that seemed to be present in David’s conversation with Colm and Reece, who just came to the table and joined the conversation. With this and the next transcript we consider how the children indexed the exterior discourses, given that they did not use the common modal verbs “have to” and “need to.”

156	David:	Do you want to count, the beans?
157	Colm:	I’ll count the beans.

158 David: [*Facing Colm*] Do you want to ask, what's, [*turning to Reece*] what's your name?
159 Reece: Reece.
160 David: Sorry?
171 Reece: Reece.
172 David: [*Turning back to Colm*] Colm, do you want to ask Reece to count some beans, Colm?
173 Colm: Well, I'm cleaning this up so it doesn't go…
174 David: Well maybe you can clean it up, and ask, and then ask Reece to count, just like I asked you. Do you want to do that? Now that's enough in here and then… There. [*Dumps beans on table*]

First, if we focus on the initial part of the conversation between David and Colm, we might interpret Colm's willingness to engage in the tasks David gave him as part of a teacher/student discourse. We see the familiar lines of David, the adult, who initiated tasks and the boys, children, who enacted the tasks. David not only asked Colm to count, but he also invited him to ask his friend to count. Colm changed the storyline and said that he would be cleaning up. With this choice he could neither invite Reece nor continue counting himself, though perhaps he was thinking that Reece should start fresh with the beans in the cup like they had been when he started talking with David himself. Still within in the expectations of a storyline of teacher/student interactions, Colm changed the storyline, and thus displayed personal latitude. David, on the other hand, confirmed Colm's wish through, "Well maybe you can clean it up". However, in the second part of the utterance, David brought Colm back to the discourse of teacher/student interaction and asked again if Colm could ask Reece to count: "and then ask Reece to count, just like I asked you. Do you want to do that?" David avoided using his personal authority in this conversation; he instead indexed a more subtle teacher/student discourse where the teacher invites the students back to the exercise at the times were they step out from the prescribed discourse. This more subtle discourse gave a semblance of personal latitude and hence a sense of student agency but still maintained David's personal authority.

In the next excerpt, continuing the conversation above, David saw an opportunity to open discussion about what counts as a bean, as we had hoped to do. Reece counted twenty-five beans from a pile and put them one-by-one into the cup. The action of putting beans in the cup when counting them may be seen as a strategy to aid Reece in keeping track of his counting, or as a strategy to prove to David that his counting had been accurate (gesturing each number with an action that can be tracked). When Reece finished, David pointed to a piece of a bean left over on the table and asked him about that one.

182 Reece: … twenty-two, twenty-three, twenty-four, twenty-five [*putting a bean in the cup for each number*].
183 David: Ok, what about this one? [*pointing to a small piece of bean*]
184 Reece: Twenty-eight.

185 David: What about this? [*pointing at a small piece of bean shell*]
186 Colm: These are just paper.
187 David: No, this isn't paper, this is part of a bean.
188 Colm: No, this is. This is [*pointing at the same small piece of bean shell*]
189 David: Is that, that's not paper.
190 Colm: That's the paper from one of the beans.
191 David: Oh, that's the part that goes around the bean. It's not really paper but I don't know what it's called.
192 Colm: I don't
193 David: So you don't think that should count?
194 Colm: No. It doesn't count.
195 David: Why not?
196 Colm: It just doesn't.

This play between subtle teacher/student discourses, conceptualizations, and terminology has us wondering what conclusions we could make about students' language repertoires in relation to their conceptual knowledge of mathematics. For example, it is not clear if this conversation was a mathematics (or counting) situation for Colm and Reece. They may have seen it as a conversation about beans (this may sound very obvious, but it is likely to appear strange in a mathematics pedagogy context). Nevertheless, Colm's assertions that the small shell/paper did not count demonstrated a language strategy—a bald assertion (i.e., without justification). Perhaps he did not know how to justify his claim or perhaps he felt the claim was so obvious that no justification was necessary; the piece in question was very small.

Similarly, the tell tale marker of discursive inevitability—"going to"—was not present in this interaction. Again, we consider the boys' other ways of more subtly referencing an exterior discourse. The strongest indicator of discursive inevitability seems to be the boys' bald assertions (as with discourse as authority) and also their counting without any suggestion that another way of counting was possible. Counting through the articulation of numbers is a unique grammatical situation because there are no verbs, and no nouns; there is only a string of words (perhaps they are adjectives, number words that refer to the objects being counted). This grammatical structure seems to leave no room for personal authority or personal latitude—no room for human agency. Yet the children in this episode demonstrated that personal latitude is possible in counting. They exercised their agency by determining what objects warranted inclusion in the count, and what objects did not qualify to be counted.

5 Discussion

Our analysis of these situations in which children were counting demonstrates the intersection of multiple discourses. From this we problematize the idea that any one of these interactions was simply a counting situation, or, more generally, simply a mathematical situation. This is despite our intentions to develop counting situations, and our aim for the conversations to focus on counting. Simply put, the children in the situations had other ideas in mind (some explicitly and others implicitly) and enacted (or at least indexed) other discourses besides mathematical or counting practices. And we ourselves had other concerns impacting our choices about how to participate in these dialogues.

We close with consideration about the potential contribution such analysis makes. For teachers, we suggest it is potentially valuable to think about the different discourses that may be in play in any classroom situation. It would be likewise helpful to think about what indicators can help identify which discourses might be at play. Such awareness may help teachers be attentive to how students are experiencing a classroom situation (e.g., students engaging in a mathematical problem solving context). This same kind of awareness ought to inform researchers who, from our experience, too readily describe what is happening in a situation without thinking about the way(s) in which students, teachers, or others in the analysed situation might be interpreting the situation differently than we are, or without including discourses other than school mathematics in the analysis.

It is also instructive for teachers to realize that students' expressions of ability and inclination are not easy to separate. In particular, assessments of mathematical ability may seem quite straightforward when we are sure what discourse a student is envisioning in the situation. However, with a different discourse in mind, the kinds of activity that could be valued may well vary. Thus, a teacher may assess students' ability on the basis of their lack of action or lack of attention, while the students does not even realize that there is a specific skill, action, or attention focus for which they are being evaluated.

Finally, the focus of our analysis points attention to the language repertoires of children as we see/hear them enacted or indexed in interaction. We suggest that it is important for teachers to be aware that students' language repertoires draw from and are at the intersection of multiple discourses. Thus there is value in encouraging the recognition that this intersection of discourses happens in the classroom, because students can develop proficiency in mathematical language practices by drawing on their repertoires of language practices in other discourses.

Acknowledgements This research was supported by the Social Sciences and Humanities Research Council of Canada, as part of a grant entitled "Students' language repertoires for investigating mathematics" (Principal Investigator: David Wagner).

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Explaining as Mathematical Discursive Practices of Navigating Through Different Epistemic Fields

Kirstin Erath

Abstract This chapter introduces a conceptualisation of explaining as mathematical discursive practices of navigating through different epistemic fields and uses this framework for analysing collective explanations in whole-class discussions. The framework coordinates Interactional Discourse Analysis from linguistics with interactionist and epistemological perspectives from mathematics education. After outlining the main ideas of the three perspectives on explaining, I describe how the notion of practices functionally links theories from linguistics and mathematics education. Furthermore, I show how the conceptualisation simultaneously highlights the interactive nature of explaining processes while also keeping the mathematical content in focus. Finally, I outline the method of identifying explaining practices in transcribed video data.

1 Introduction: The Importance of the Discursive Level of Language in Mathematics Education

The interplay between language and learning mathematics is a growing area in mathematics education research (e.g., Barwell et al., 2016): Many researchers are working on the questions about how this interplay works (descriptive perspectives), how students should communicate in mathematics classrooms (normative perspectives), and how teachers and learning arrangements can support students to facilitate mathematical learning opportunities through language (developmental perspectives). More and more studies in mathematics education point to the importance of the discursive level of (spoken) language for the meaningful learning of mathematics for all children, and especially for those with emerging language proficiency (e.g., Barwell, 2012; Erath, Prediger, Heller, & Quasthoff, in review; Moschkovich, 2002, 2015). These studies can be roughly subsumed under the term *participationist perspectives* (Krummheuer, 2011; Sfard, 2008) since they are

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© Springer International Publishing AG, part of Springer Nature 2018
J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_10

united under the assumption that participation in classroom interaction is fundamental for learning mathematics. This study shares this participationist perspective and aims to further conceptualise how participation in classroom discourse is connected to (unequal) opportunities to learn mathematics.

Questions about how discourse can be conceptualised in order to examine different aspects of classroom interaction (e.g., positioning, language usage, mathematical ideas, strategies for problem solving, etc.) and how discourse impacts mathematics learning are not completely answered yet. For example, Gee's (1996) conceptualisation distinguishing between 'discourse' and 'Discourse' is often used in mathematics education (e.g., Moschkovich, 2002, 2013, 2015): Building on Gee, Moschkovich (2002, p. 198) defines mathematical Discourses as including "mathematical values, beliefs, and points of view of a situation". This conceptualisation emphasises socio-cultural aspects but may not directly focus on the discussed mathematical topic and does not broach the issue of what language resources students need for successful participation in discourse. The role of the discussed mathematical topic and language proficiency are two other possible foci of a conceptualisation of discourse in mathematics classrooms (amongst others). The theoretical framework presented here focuses on directly embedding the discussed mathematical topic in a conceptualisation of explaining in whole-class discussions—while also keeping the interactive nature of explaining in mind—and therefore offers another perspective on the interplay between the discursive level of language and the learning of mathematics.

Explaining was chosen out of the various possibilities of linguistically distinguishing discourse (e.g., arguing, narrating, describing, reporting) since it appeared most often in a study on German grade 5 whole-class discussions (Erath et al., in review) and can be shown as important for the conceptual and meaningful procedural learning of mathematics (Erath, 2017a). Furthermore, only whole-class discussions were considered, because the study aims at further investigating the role of explaining during collective knowledge construction.

The chapter addresses the following research questions:

Q1 How can perspectives from linguistics and mathematics education be coordinated for conceptualising oral explanations in mathematics classrooms?
Q2 How can the mathematical aspects of a discussion be addressed in a conceptualisation of collective explanations?

These questions are answered by introducing the conceptualisation of explaining as *mathematical discursive practices of navigating through different epistemic fields* (Sect. 2.4) that considers the interactive nature of collective explanations and at the same time focuses on the mathematics being discussed. This conceptualisation was developed as part of my PhD-thesis (Erath, 2017a) describing collective explanations in German grade 5 mathematics classrooms. That study occurred in the context of a larger project, INTERPASS, which focused on the analysis of the micro level of interaction in whole-class discussions. The proposed framework is based on the linguistic perspective of Interactional Discourse Analysis (Sect. 2.1) which offers a definition of explaining that is compatible with the interactionist perspective from mathematics education (Sect. 2.2). I outline how the notion of practices serves as a link between the two perspectives. The question about possible topics of explanations in mathematics and by which means they can be explained is answered by merging studies from an epistemological perspective in mathematics education in Sect. 2.3. I present short examples of the process of identifying practices by means of the introduced conceptualisation in Sect. 2.5. As outlined in Sect. 3, the proposed theoretical framework was already successfully used for a descriptive analysis of explanations in whole-class discussions in mathematics classrooms, to characterise students' participation, and in collaborative research with colleagues from linguistics.

2 Theoretical Perspectives: Conceptualisation of Explaining Practices in Whole-Class Discussions

2.1 Explaining in Interactional Discourse Analysis

As a first step, we need to define what is meant by the notion of *explaining in general*. I use the linguistic theory of Interactional Discourse Analysis (IDA; Quasthoff, Heller, & Morek, 2017) as a starting point, since it is used to analyse oral communication at the micro level of interaction which is also the unit of analysis of the underlying study of this chapter (Erath, 2017a).

In IDA, explaining is defined as a *discourse practice* (Morek, Heller, & Quasthoff, 2017, p. 17, translated by the author), "that is

- realised in a specific context ("contextualised"),
- interactively produced and co-constructed, and
- made recognisable for each other as explaining […] or rather 'accountable' (Garfinkel, 1967)".

'Contextualised' means that explaining is always embedded in a specific course of conversation and a social context (e.g., the whole-class discussion in a particular mathematics classroom) and at the same time explanations shape this social context. Furthermore, all involved persons participate in producing the discourse practice—speakers and listeners. The conceptualisation of explaining as discourse practice also implies that explaining is a routinized way of solving a recurrent communicative problem in a speech community, i.e., discourse practices are *genre-oriented*, using the concept of genre from the sociology of knowledge (Bergmann & Luckmann, 1995).

This definition also allows us to distinguish between explaining, arguing, reporting, narrating, etc. by discerning the different communicative problems these solve (Bergmann & Luckmann, 1995; Quasthoff et al., 2017): For example, explaining solves the recurring communicative problem of constructing and demonstrating knowledge, whereas arguing solves the communicative problem of divergent validity claims (Morek et al., 2017). In addition, IDA's definition of explaining indicates how to identify explanations in whole-class discussions: a sequence is identified as explaining, if there is a common focus of attention that is treated as in need of an explanation concerning what, why or how (Morek, 2012, p. 40) and—in delimitation to arguing—if the validity of the discussed matter is not up for debate through the whole sequence.

From the perspective of IDA, explaining in whole-class discussions in mathematics classrooms is a discursive practice that serves to construct and demonstrate knowledge and by this function differs from other discourse practices. Furthermore, it can be expected that explanations vary in different classrooms and are different from explanations at home since they are realised in different contexts (e.g., teacher A with 30 students and teacher B with 30 other students; a family discussing during dinner). Therefore, this linguistic perspective offers a definition of explaining as a collective and interactive process that is closely tied to its context. But naturally it does not include content specific aspects of explaining in *mathematics* classrooms. In order to grasp these aspects, two perspectives on explaining from mathematics education are presented in the following sections. An interactionist perspective that serves as a link to IDA and allows to further specify the context of explanations (Sect. 2.2) and an epistemological perspective that addresses the mathematics that is explained (Sect. 2.3).

2.2 Explaining from an Interactionist Perspective in Mathematics Education

The interactionist perspective in mathematics education focuses on the same micro level of interaction as IDA and has established the concepts *classroom microculture*, *sociomathematical norms*, and *mathematical practices* (Cobb & Bauersfeld, 1995; Cobb, Stephan, McClain, & Gravemeijer, 2001; Yackel & Cobb, 1996). The concept of classroom microculture is used for further conceptualising the notion of context from IDA and has a special position in interactionism since it "is neither ignored nor taken for granted: it is the most basic object of study" (Sierpinska & Lerman, 1996, p. 853). Nickson's definition emphasises that every class establishes its own microculture with each teacher:

> What has emerged in this study is that the culture of the mathematics classroom will vary according to the actors within it. The unique culture of each classroom is the product of what the teacher and pupils bring to it in terms of knowledge, beliefs, and values, and how these affect the social interactions within that context. (Nickson, 1992, p. 111)

Furthermore, culture is understood as comprising all patterns of action, interpretation, and perception that are established in a social community, e.g., a mathematics classroom.

Sociomathematical norms (Yackel & Cobb, 1996) and mathematical practices (Cobb, 1998; Cobb et al., 2001) are two constructs for analysing classroom microcultures. In the following, the chapter focuses on mathematical practices that were established as a "theoretical construct that allows us to talk explicitly about collective mathematical development" (Cobb, 1998, p. 34). Following Cobb et al. (2001), mathematical practices are not understood as already existing ways of reasoning or communicating that students need to be introduced to. Instead, they are interactively established in a classroom microculture and serve to describe changes in the collective mathematical action and communication of the whole microculture (for a slightly different conceptualisation of mathematical practices from a Vygotskian perspective see Moschkovich, 2013).

The construct has its origin in developmental research (Cobb, 1998; Cobb et al., 2001) as way of planning intended learning trajectories as assembled mathematical practices including the means of supporting their establishment. In a second step, the construct was used for analysing episodes from classrooms in order to identify the actual learning pathways of the class. In the conceptualisation of explaining as a practice presented in this chapter, mathematical practices are only considered from a descriptive perspective, i.e., for analysing episodes of classroom interaction without planning or implementing particular practices in advance from a normative perspective.

A second point of reference is the work of Kolbe, Reh, Fritzsche, Idel, and Rabenstein (2008); although this approach is not part of the interactionist perspective from mathematics education, their conceptualisation of a culture of learning from general educational science can be connected to the concept of practices. They understand practices as "rule-governed, typecasted, and routinely recurring activities" (Kolbe et al., 2008, p. 131; translated from German by the author) and thus as a phenomenon that can be ethnographically observed and identified. In their conceptualisation, "rule-governed" means that practices have structures with a certain orderliness that can be identified in the interaction. These structured activities are typecasted since different practices have different underlying structures which are a means of distinguishing them. Furthermore, activities are only identified as practices if the underlying structure can be repeatedly observed in the interaction.

Altogether, mathematical practices are defined in this chapter as recurring, interactively established courses of action (in a very broad sense) in a classroom microculture that can be identified by characteristic underlying structures. That is, explaining can be conceptualised as a mathematical practice as follows: explanations are interactively established ways of communicating that can be repeatedly observed and have a specific structure that makes them distinguishable from arguments, descriptions, reports or narrations.

The question of what counts as an explanation cannot be answered in general. From an interactionist perspective, there is no generally right or wrong explanation referring to an external normative mathematical perspective but explanations match or do not match the established practices and sociomathematical norms:

> A further implication of treating explanation, justification and argumentation as interactional accomplishments is that detailed analyses focus on what the participants take as acceptable, individually and collectively, and not whether an argument might be considered valid from a mathematical point of view. (Yackel, 2004, p. 3)

Already existing research mainly describes identified sociomathematical norms for explanations as well as various processes for establishing those norms (Yackel, 2004; Yackel & Cobb, 1996; Yackel, Rasmussen, & King, 2000). The study summarized here builds on this work but aims at identifying the practices of explaining regarding the topics and means of explanations as manifestations of sociomathematical norms in order to describe mathematical explanations (as a learning goal in mathematics classrooms) more precisely.

2.3 *Explaining from an Epistemological Perspective in Mathematics Education*

As a third step, I consider an epistemological perspective from mathematics education on explaining in order to grasp the mathematics of the explanations and the role of explanations in the collective processes of knowledge construction. This research has a normative perspective on what counts as a good explanation or how students should explain if they have completely understood a specific topic. In the conceptualisation of explaining introduced in this chapter, this normative perspective is used to collect potential topics and means of explaining and systemises them in an *epistemic matrix* (see Fig. 1; Erath, 2017a; Prediger & Erath, 2014)

Epistemic modes / Logical levels	Labelling & naming	Explicit formulation	Exemplification	Meaning & connection	Purpose & evaluation
Conceptual levels					
Concepts					
Propositions					
Semiotic representations					
Models					
Procedural levels					
Conventional rules					
Procedures					
Concrete solutions					

Fig. 1 Systematisation of topics (logical levels) and means (epistemic modes) of explanations in the epistemic matrix (Erath, 2017a; Prediger & Erath, 2014)

which is used (1) to complement the conceptualisation of explaining in mathematics classrooms (see Sect. 2.4) and (2) to identify explaining practices and analyse students' participation in explaining from a descriptive perspective (see Sect. 3).

From an epistemological perspective, there are various aspects of mathematical knowledge students should acquire in school and therefore should also be able to explain. The differentiation between conceptual and procedural knowledge (Hiebert, 1986) is expanded in this study by work on general taxonomies (Anderson et al., 2001) and taxonomies specific for the learning of mathematics (Vollrath, 2001) as well as work specifying the mathematical knowledge that should be addressed in the phases of knowledge organisation and systematisation in mathematics classrooms (Barzel, Leuders, Prediger, & Hußmann, 2013). Findings from these studies were collected and enhanced by categories derived from the analysis of the video data (by means of inductive qualitative content analysis as in Mayring, 2015; see Erath, 2017a) in the logical levels (rows) of the epistemic matrix in Fig. 1.

The four conceptual levels comprise

- concepts (e.g., prime numbers, perpendicular bisector, or linear function),
- propositions (e.g., Pythagorean theorem, but also mathematical patterns and statements in a broader sense),
- semiotic representations (e.g., a chart on the blackboard, but also all verbal, gestural, and graphic realisations of a mathematical topic or idea), and
- models (addressing the transition from reality and mathematics; e.g., Which geometrical field describes a hot air balloon?).

The three procedural levels are

- conventional rules (e.g., rules for rounding whole numbers; rules that are plausible but not deduced from theorems or concepts)
- procedures (e.g., calculation of the average or Gauss-Jordan method), and
- concrete solutions as individual ways of solving a specific task.

Whereas the logical levels are categories for possible topics of explanations in mathematics classrooms, the epistemic modes in the columns of the epistemic matrix answer the question of what means can be used to explain a mathematical idea or concept. These categories are derived from research on facets of concepts (Winter, 1983), pragmatic functional approaches to the learning of mathematics (Brousseau, 1997; Freudenthal, 1983; Wagenschein, 1968), mental models (vom Hofe, Kleine, Blum, & Pekrun, 2005), and two more comprehensive approaches already mentioned above (Barzel et al., 2013; Vollrath, 2001). Like the categories for logical levels, the theoretically derived categories were complimented and enhanced by an inductive qualitative content analysis (Mayring, 2015) in Erath (2017a).

The collection of epistemic modes as different means of explaining a mathematical aspect are:

- labelling and naming (e.g., stating a number as solution of a calculation or the name of a geometric field),
- explicit formulation (e.g., defining a concept or formulating theorems in general, i.e. without referring to examples),
- exemplification (e.g., giving examples and counterexamples for a concept)
- meaning and connection (e.g., mental models, referring to pre-existing knowledge in the meaning of already known mathematics like explaining the concept of maximum by referring to the (known) concept of minimum), and
- purpose and evaluation (e.g., constraints of applying an algorithm or discussing which representation is handier for solving a problem).

The epistemic matrix offers the possibility to characterise every requested or given explanation in whole-class discussions in mathematics classrooms by identifying which logical level is addressed (which content needs to be explained/is explained) and by assigning the utterance to epistemic modes (by which means is the content to be explained/is explained). That is, every utterance of an explaining episode can be characterised by the so-called epistemic fields (the cells of the matrix as combinations of logical levels and epistemic modes). And—as will be outlined at the end of the next section—it is also possible to mathematically characterise whole episodes of explanations and describe their role in the process of knowledge construction.

From an epistemological perspective, explanations in mathematics classrooms should include several aspects (e.g., Winter, 1983). In the language of the epistemic matrix this means that an explanation should address several epistemic fields. On the one hand, this shows the understanding of the students (e.g., Vollrath, 2001), on the other hand the connection between different logical levels is also important (for an example, see the analysis in Hiebert, 1986). Furthermore, it can be anticipated that collective explanations in whole-class discussions consist of utterances from several students and that the teacher points to different epistemic fields in order to explain a specific topic. From an epistemological perspective, the teacher has a special position in the process of explaining, since he or she has the responsibility for the epistemic process the explanation is based on. Therefore, it is to be expected that the teacher will navigate an explanation (didactically and pedagogically motivated) through different epistemic fields and influences the process especially by evaluating explanations as matching or not matching his/her epistemic expectations.

2.4 Synthesis: Conceptualisation of Explaining as Multiple Practices of Navigating Through Different Epistemic Fields

In this section, the three presented theoretical perspectives are coordinated in order to grasp mathematical and linguistic aspects of collective explaining in mathematics classrooms. In the framework of networking theories (Prediger, Bikner-Ahsbahs, & Arzarello, 2008, p. 172), the term *coordinating* is used “when a conceptual framework is built by well-fitting elements from different theories”. First, I will show how Interactional Discourse Analysis and the interactionist perspective from mathematics education are coordinated by means of the notion of practices and on the basis of their shared methodological roots. In a second step, I will show how the epistemological perspective is coordinated with the two perspectives focusing on interaction.

Referring to Interactional Discourse Analysis (Quasthoff et al., 2017), explaining is conceptualised as an interactively established discursive practice (Sect. 2.1). Explaining must be considered as embedded in a social and instructional context, since this contextualisation distinguishes explaining in classrooms from explaining in families and also explaining in classrooms with different teachers. Explaining has a special position in whole-class discussions since this discourse practice is required for constructing and demonstrating knowledge.

From an interactionist perspective (Sect. 2.2), explaining is conceptualised as a mathematical practice that is interactively established in mathematics classroom microcultures. Also from this perspective, we expect that structurally similar explanations can be recurrently observed in the classroom interactions. The notion of microculture, as a central point of reference, also implies that explanations are established and arranged differently in different classrooms. For example, it can be matching one microculture to explain a concept by examples, whereas another microculture might require explaining a concept by formulating a general definition.

The interactionist perspective from mathematics education and IDA complement each other, this can be traced back to the shared roots of symbolic interactionism (Blumer, 1969) and ethnomethodology (Garfinkel, 1967). The concept of practices can be seen as a link between these two perspectives: Both emphasise that practices are interactively established and closely linked to the context in which they are enacted. The social context from IDA is taken on by the concept of microculture from the interactionist perspective in mathematics education and further refined for mathematics classrooms by the notions of sociomathematical norms and mathematical practices. In addition, both perspectives conceptualise practices as recurring ways of acting. From both perspectives it is to be expected that structurally similar explanations are recurrently established as matching in mathematics whole-class discussions in ways that are specific to each microculture.

The coordination of IDA and the interactionist perspective from mathematics education also reveals the significant challenges that students and teachers face in whole-class discussions. On the one hand, explaining is a medium for learning mathematics, since it serves the construction and demonstration of knowledge. On the other hand, explaining is a mathematical learning goal since mathematical explaining is a practice mainly acquired in classrooms (Quasthoff & Heller, 2014) and differs from explaining at home (Sect. 2.1). In addition, explaining is also a linguistic learning goal. All students in secondary school (aged around 10–15 years) are still in the process of acquiring discourse practices in general and not all students share the same experiences of explaining, arguing etc. in the context of family and/or peer groups (Quasthoff & Morek, 2015). The conceptualisation of explaining as mathematical discursive practices highlights this parallelism of language and mathematics and mirrors the inseparability of linguistic and mathematical learning which is often emphasised in mathematics education (e.g., Barwell, 2012; Lampert & Cobb, 2003; Prediger, 2013).

The coordination of the two perspectives focusing on interaction and the epistemological perspective from mathematics education allows us to mathematically characterise the expected structural similarities of interactively established explaining practices. Using the epistemic fields in the epistemic matrix, student and teacher utterances can be characterised mathematically and in their role in knowledge construction. At the same time, using the epistemological perspective highlights the importance of explaining for learning mathematics, since connecting a logical level with an epistemic mode can make the networked structure of mathematical knowledge explicit (Erath, 2017a).

Altogether, these theoretical perspectives lead us to expect ‘explaining pathways’ through the epistemic fields that students and teachers will jointly follow in classroom interaction. Each utterance addressing one or more epistemic fields is a ‘footprint’. But it is especially the teacher who navigates, didactically motivated and in relation to the utterances of the students, through the epistemic matrix and thus gives a ‘direction’ for the subsequent ‘footprints’ (students’ utterances). The idea of collective explaining pathways mirrors the idea of an interactive production and co-construction (perspective of Interactional Discourse Analysis) and the interactive establishment in a microculture (interactionist perspective in mathematics education) since contributions from students and teachers are not seen as separate but as always referring to one another. Another important aspect from mathematics education is concretised by the idea of explaining pathways, we can observe how students and teachers address different epistemic fields in the process of explaining a mathematical topic. These steps are mainly not constituted by ‘false steps’ from students (for empirical evidence see Erath, 2017a) but result from the didactically motivated navigations of the teacher, who purposefully uses the steps in order to process the mathematical content for the students.

As elucidated above in the coordination of IDA and the interactionist perspective in mathematics education, we can expect that structurally similar explaining pathways are recurrently established as matching explanations specifically for each mathematics classroom’s microculture. These groups of similar explaining pathways constitute the explaining practices in a microculture (for more information concerning methodology and detailed empirical evidence see Erath, 2017a). Therefore, explaining is conceptualised as mathematical discursive practices of navigating through different epistemic fields, i.e. as multiple practices of jointly treading recurrent and interactively established pathways through an epistemic matrix. In this way, explaining is conceptualised as an interactive process while simultaneously and explicitly taking into account the mathematical content.

2.5 *Insights into the Process of Identifying Explaining Practices*

The presented conceptualisation of explaining as mathematical discursive practices of navigating through different epistemic fields (short: explaining practices) implies a closely related method for identifying explaining practices in whole-class discussions (developed in Erath, 2017a):

(1) Each episode of explaining is analysed by means of the categories of the epistemic fields. That is, each student or teacher utterance is characterised by one or several epistemic fields.
(2) The results of the analysis in (1) are displayed by representing each utterance as an entry in a rectangle (for students, including name and turn number) or circle (for the teacher, including turn number) in the epistemic fields it is assigned to. Furthermore, the teacher's navigations are made explicit by arrows from the last epistemic fields that were addressed by a student to the epistemic fields the teacher addresses in his requests for the next utterance. Thus, each episode is concentrated in a picture of its explaining pathway as exemplified in Fig. 2 for the episode "the meaning of rounded zero" (cf. Erath et al., in review).
(3) Steps (1) and (2) are repeated for all explaining episodes observed in a classroom microculture.
(4) The actual step of identifying practices directly refers to the definition of mathematical practices as rule-governed, typecasted and routinely recurring (Sect. 2.2): Explaining pathways with similar structures concerning addressed epistemic fields or steering activities from the teacher are grouped by means of inductive qualitative content analysis (Mayring, 2015). Here, the specific feature of each group serves to characterise the different practices of explaining evident in the observed microculture.

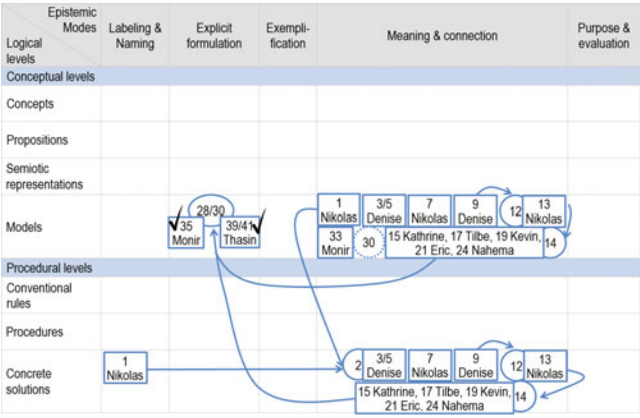

Fig. 2 Explaining pathway of the episode "meaning of rounded zero" (see Erath et al., in review for the corresponding analysis)

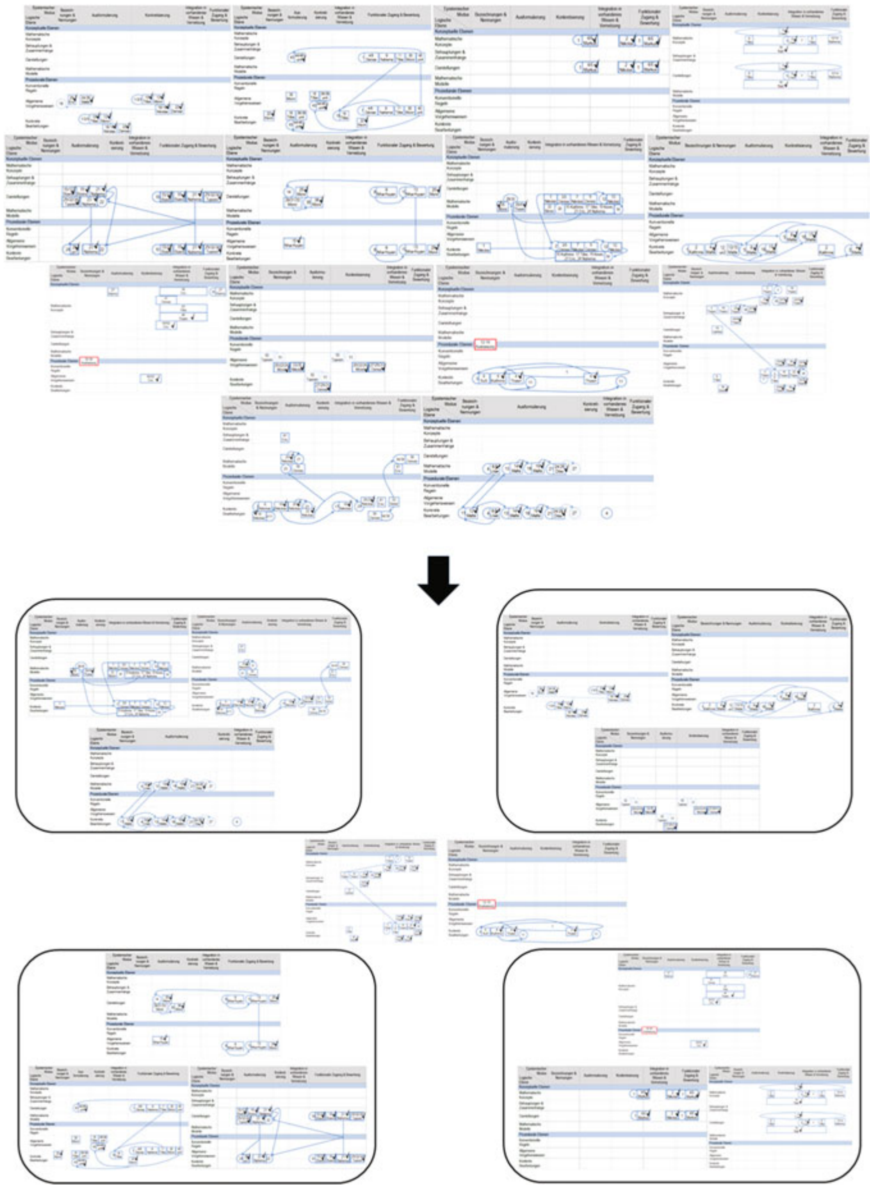

Fig. 3 Exemplification of step (4) in the process of identifying explaining practices in a mathematics classroom microculture (Erath, 2017a, p. 83)

Figure 3 (which is not meant to be readable but to show similar structures of explaining pathways) exemplifies step (4) by means of 14 out of 31 explaining episodes in Mr. Schrödinger's classroom (Erath, 2017a). The explaining pathways of the episodes are displayed chronologically in the upper part of the figure and the result of the process of building categories at the bottom of the figure. The frames unite pathways that together constitute one practice of explaining. For example, the three pathways in the rectangle at the bottom left are part of the practice of explaining 'good' representations in the microculture of Mr. Schrödinger's classroom. All pathways contributing to this practice have in common that they have entries in the rows of the logical levels 'semiotic representations', 'procedures', and 'concrete solutions' and that after working in the epistemic mode 'purpose and evaluation' (more on the right) the teacher navigates to the mode 'explicit formulation' (more on the left) which is indicated by arrows (there are only lines visible in Fig. 3 because of the restricted size but there are arrows in the original). These structural features are characteristic for this group of pathways and makes them distinguishable from the other groups of pathways. Furthermore, pathways with these features can be repeatedly described also across several lessons (in this case five episodes in nine consecutive lessons) which means that they are routinely recurring. The role of explanations in this particular explaining practice in the process of knowledge construction can also be derived from the characteristic features. Explanations started with student (or the teacher) describing what are "good or not yet so good" features of a representation (e.g. a diagram) as a solution of a task given by the teacher. Next, the teacher asked for ways of improving the representation and then navigated to asking for explicit formulations of the characteristics of 'good' representations (the epistemic mode). Thus, each sequence includes a process of abstraction from a specific example to more general characteristics.

3 Conclusion

The first research question, how perspectives from linguistics and mathematics education can be coordinated for conceptualising oral explanations in mathematics classrooms, was answered by showing that the concept of practices is a functional link between Interactional Discourse Analysis and an interactional perspective in mathematics education. This was shown for the case of explaining but this approach has the potential to be applied for other discourse practices, such as arguing or proving. The second research question, how mathematical aspects can be addressed in a conceptualisation of collective explanations, was answered by introducing the epistemic matrix as a systematisation of research from an epistemological perspective in mathematics education.

Altogether, the three theoretical perspectives on explaining in whole-class discussions highlight different aspects that are all fundamental for understanding oral explaining in mathematics classrooms. At the same time, the perspectives are not juxtaposed but carefully coordinated, which facilitates the pursued understanding of explaining as a simultaneously linguistic and mathematical phenomenon in mathematics classrooms.

Erath (2017a) offers a detailed analysis of explaining practices in four classroom microcultures (in German). That analysis shows that the introduced conceptualisation of explaining and the derived method of identifying explaining practices works and allows the analyst to (1) mathematically characterise single utterances but especially whole sequences by means of the epistemic fields and the practices of navigating through different epistemic fields as well as (2) grasping their role in the process of knowledge construction. As also shown in Erath and Prediger (2014) and Prediger and Erath (2014), what counts as a good explanation and how explanations are organised depends strongly on the classroom microculture. This result leads to the question of how students learn to adequately participate in the explaining practices of their particular classroom microculture. As outlined in Erath (2017b), the processes of establishing practices are mainly implicit, which means that the teacher may not make his criteria for evaluating student explanations explicit. Thus, learning how to successfully participate in the interactively established explaining practices may be challenging for all students.

Another way of using the epistemic matrix is the description of individual student participation in explaining episodes. The epistemic matrix offers a language for talking about different ways of participating in explanations and how these are connected to the related processes of knowledge construction. This is exemplified for the case of three students from Mr. Schrödinger's classroom in Erath and Prediger (2015): Locating the students' utterances in the epistemic matrix facilitates a description of different epistemic participation profiles for each student.

Besides these studies from researchers in mathematics education, the presented conceptualisation also allows further collaboration with researchers from linguistics. For example, Erath et al. (in review) build on and further explicate the coordination of IDA and mathematics education perspectives by working on the connection between discourse competence and mathematics learning. This analysis contributes to an empirically grounded conceptualisation of academic language proficiency at the discourse level and thus builds a foundation for further research on supporting students in participating in collective explaining and other discourse practices in mathematics classrooms.

Grant Information The research project INTERPASS (Interactive procedures for establishing matches and divergences in linguistic and microcultural practices) is funded by the German ministry BMBF (grant 01JC1112, grant holder S. Prediger). I have conducted it under the guidance of Susanne Prediger and Uta Quasthoff, together with Anna-Marietha Vogler and Vivien Heller.

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Part III
Teacher Focused Research

Interactional Processes in Inclusive Mathematics Teaching

Judith Jung

Abstract The UN Convention on the Rights of Persons with Disabilities requires that persons with disabilities should not be excluded from the general education system. This legal right surpasses the mere freedom of choice to attend a regular school. Against the backdrop of this transformation of the practice of mathematics learning in Germany, this article examines the potential for social and content-related participation in inclusive school teaching that may emerge in whole class discussions among students, teachers and pedagogical staff. The data comprises transcripts of video-recorded lessons that were analysed through interactional analysis. The presented analyses of classroom conversations in a Year 1 class reveal interaction that was structured by the teacher through repetitive sequences, enabling the participation of many pupils but providing them few opportunities to participate outside of these structures.

1 Starting Point: Inclusion and the German School System

In Germany, most children first attend primary school from the age of five/six to nine/ten and are then separated by an achievement-based selection process into different secondary school forms—this is a system that has long been criticised in several respects. Furthermore, from the mid-18th century, a parallel branch of schooling for children with special educational needs has been established along-side primary school and the four types of secondary school—the special needs school. The special needs school system continued developing into the 2000s, with different schools specialising in particular needs, e.g. emotional and social development, learning difficulties, speech development, visual impairment, hearing impairment, intellectual development. In the 2000 school year in Germany, a total of 479,940 pupils with special educational needs went through this system. Of these, 420,587 (87.63%) were schooled at special needs schools and 59,353

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J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_11

(12.37%) in regular schools (Secretariat of the Standing Conference of the Ministers of Education and Cultural Affairs of the Länder, 2016).

With the EU's ratification of the UN Convention on the Rights of Persons with Disabilities, this praxis of the separated school system came under renewed criticism and its legality came into question. The UN Convention requires that persons with disabilities should not be excluded from the general education system because of their disability (Article 24.2). This legal right surpasses the mere freedom of choice to attend a regular school. According to the UN Convention, adequate arrangements have to be made within regular schools in order to ensure the educational success of each individual. The ratification of the Convention in Germany opened a political and social discussion on structural changes to the school system. While the German school system had hitherto been characterised by strong mechanisms of selection, seeking to ensure that children and young people with disabilities were schooled in special institutions appropriate for them, in many ways what is now required is precisely the opposite. Since 2009 a policy of inclusion—in the sense of integrative schooling of pupils with special educational needs in the regular school system—has been implemented in the German states. Thus, in the 2014 school year, of the 508,386 pupils with special educational needs who went through German schools, 334,994 (65.89%) attended a special needs school and 173,392 (34.11%) a regular school. The ratification of the Convention therefore appears to have led to a thorough implementation process, as the number of children with special educational needs in regular schools has doubled. However, looking at the figures against the backdrop of demographic developments, which have seen a reduction in pupil numbers, the proportion of pupils with special educational needs in the total number of pupils in Germany clearly rose between 2000 and 2014 (Secretariat of the Standing Conference of the Ministers of Education and Cultural Affairs of the Länder, 2016).

Nevertheless, clear—if often very hesitant—efforts have been made in individual German states towards restructuring a multiple-level school system with strict separation between regular and special needs schools towards a more and more open, inclusive school system. However, only one of the five biggest Federal states, which together contribute around 70% of the German population, achieved a proportion of above 25% (Klemm, 2013). It seems there is still a long way to go to achieve a school system that can be described as inclusive, although current efforts will tend to lead to a well-constructed integrative school system, since a really complete system of inclusion would need a dismantling of the categorisation of children not corresponding to the presumed norms. We follow Katzenbach (2012) in holding that there is little difference between the terms integrative and inclusive in school life. However, there are some clear conceptual differences: the idea of integration depends on a categorisation of people. There are "normal" people and "the others", the non-disabled and the disabled, and these need to be brought together. The concept of inclusion, however, is based on the intellectual premise of diversity. Disability is only one characteristic among many, and school is a place where people with extremely different characteristics intermingle. Following

UNESCO's (2005) understanding of the concept, inclusion can be understood as an on-going process to find better ways of responding to diversity.

Towards providing an empirical base for this process of changing a traditionally strongly separated school system, like that in Germany, in respect to content-related—here mathematical—learning conditions, this paper discusses some theoretical considerations on mathematical learning and learning under inclusive conditions, and synergises these for an empirical analysis. The coordination creates a conceptual framework that helps identifying students' participation in inclusive settings and learning in mathematics.

2 Theoretical and Methodological Foundations

Diverse efforts in mathematics education research are now focusing on investigating different aspects of opportunities for collective and individual mathematics learning in inclusive settings. This study focuses on investigating interactions in inclusive mathematics teaching, under the following basic assumptions. Mathematics learning is understood as in social-constructivist and sociological theories of learning, as an interactive process that, above all in the early years, occurs through participation in content-related processes of negotiation with others (Miller, 1986). From a social-constructivist perspective, learning cannot be seen as a primarily internal cognitive restructuring process. Rather, it is a dualistic process that takes place both within the individual, in the sense of cognitive restructuring, and within interaction processes in which the person participates, which go before these restructurings. Sfard (2008, p. xxi) elaborates: "once we agree that thinking is an individualized form of interpersonal communication, we must […] concede that whatever one creates is a product of collective doing."

This kind of sociological or social-constructivist consideration of learning processes has in recent years gained increasing influence in the theoretical design of content-related learning, and has been taken up and further developed in mathematics education research (Lerman, 2000). Both nationally and internationally, mathematics has increasingly come to be seen as a cultural tool, constructed and mediated through language (Schütte, 2014). Since the mid-1980s, interactionist approaches to interpretive (classroom) research in mathematics education have engaged with the sociologically based social-constructivist perspective on learning processes (e.g. Bauersfeld, Krummheuer & Voigt, 1988) under theories of symbolic interactionism (Blumer, 1969). With this kind of basic theoretical understanding of content-related learning, the concept of collective argumentations gained central significance in the analysis of mathematical learning processes. According to Krummheuer and Brandt (2001, see also Krummheuer, 1995), pupils are usually engaged in interaction processes in classroom conversations, producing an argumentation in the totality of their actions. In this way, participation in a collective argumentation concerning statements about (mathematical) content, terms and/or methods creates the basic conditions for learning opportunities. This interplay of individual and social constituents is difficult to describe. If participation in collective argumentation provides orientation and convergence, then learning success can be seen as the improved coordination between individual attributions of meaning and the results of the interactive negotiation of meaning in the respective group. On an interactional level, this is manifested in an increasing adaption of the (verbal) acts of the learners to argumentations established collectively over the course of several interactional situations. The coordination of an individual's interpretations and actions can be reconstructed empirically as the increasingly autonomous adoption of steps of action within the collective argumentation. The learning of mathematics can thus be described as the "progress" of participation in mathematical collective argumentations. This idea of learning through participation can be linked back to the notion of equal opportunities for participation in educational institutions, according to the UN Convention on the Rights of Persons with Disabilities.

Among the goals set by the increasing implementation of the inclusion concept in teaching, providing all pupils with equal opportunities for participation in learning processes occupies a central position. This reflects UNESCO's principle whereby "[i]nclusion is seen as a process of addressing and responding to the diversity of needs of all learners through increasing participation in learning, cultures and communities, and reducing exclusion within and from education" (UNESCO, 2005, p. 13). This principle, formulated in the framework of the UNESCO "Guidelines for Inclusion", is reflected in the UN Convention on the Rights of Persons with Disabilities, ratified by Germany in 2007. The Convention states: "Persons with disabilities are not excluded from the general education system on the basis of disability. Persons with disabilities can access an inclusive, quality and free primary education and secondary education on an equal basis with others in the communities in which they live" (Article 24, 2a+b). In addition, the Convention asserts that children/persons with disabilities should be supported within general education, according to their needs, to best progress their education. A "full and equal participation in education" (Article 24.3) should be made possible for them. A few years back, Booth and Ainscow (2002) already published an "Index of Inclusion", which is intended as a tool to support an inclusive school development and contains a detailed description of how barriers to learning and participation for all learners can be dismantled. Here, too, the goal of a "greater participation of students in the cultures, curricula and communities of their schools" is cited (Booth & Ainscow, 2002, p. 2). In summary, according to the mentioned literature, inclusion can be understood as an unending process of increasing learning and participation for all students (Booth & Ainscow, 2002; UNESCO, 2005) and thus also as an ideal, which will never be fully realised but is already applied with the start of the process of increasing participation. However, it seems necessary to clarify at this point what is understood by full and equal opportunities for participation. Therefore, the concept of participation in learning processes is developed in the following. According to Booth and Ainscow (2002, p. 3), participation can be understood as "learning alongside and in collaboration with others in shared learning experiences". They see participation as demanding active involvement in learning processes and the opportunity to express one's own learning experiences.

In an empirical investigation of teacher communication in inclusive mathematics lessons in US schools, Wiebe Berry and Kim (2008) indicated that inclusive settings are usually defined above all by the narrowness of the teaching, with pupils given few stimulations for use of language or mathematical argumentation. Teachers mainly pose questions whose answers are known in advance, supporting the "traditional" format of the teacher's role. Such "small-steps" questions have the advantage that many children can answer, but without using many words or making mistakes, which ensures that the teaching can continue uninterrupted. However, this kind of communication does not appear appropriate to support either active-discovery constructive learning or learning through participation in collective argumentation.

3 Research Project

The research project focused on describing examples of the process of progressing inclusion in German schools, and on this basis to design steps to further this process. The focus was on the reconstruction of interactional processes, rather than on developing direct implementations to advance inclusive teaching. This focus emerged from the assumption that research approaches seeking to understand and describe are necessary to identify potential actions based on changing conditions in schools and teaching: "If subject educators are to (…) react to changed social conditions for school learning, they need the support of sociological perspectives to allow them to appropriately describe and conceptualise these learning processes in relation to the changed social experiences of the pupils" (Krummheuer, 1992, p. 4, translated by the author). Inclusive learning conditions are taken here to have an important influence on the interactive relationships between teacher and learners and among learners. In the implementation of the full research project, video-recordings of everyday inclusive mathematics teaching in different year groups (Years 1, 2, 4 and 6) in Germany will be made and transcribed over a period of 1.5 years in order to enable the application of methods of interpretive classroom research (e.g. Krummheuer, 1992).

4 Methodological Procedure of Data Analysis

For this paper, the classroom conversations of four lessons in two Year 1 classes were comprehensively transcribed and evaluated using interactional analysis (Krummheuer, 1992). The interactional analysis enables a reconstruction of how the negotiations of meaning engaged in by individuals in the interaction are constituted and become shared interpretations. It serves to reconstruct patterns and structures in the interactional processes and thus also to describe the conditions facilitating content-related and social learning. The following research questions are at the centre: *What patterns and structures can be reconstructed in the classroom conversations? Which opportunities for participation are created for different children (within these patterns and structures)?* Towards answering the first question, descriptive systems were used that show the regularities in short sequences from everyday mathematics teaching, examining emerging patterns of interaction (Voigt, 1984) and forms of argumentation (Krummheuer, 1992). Patterns of interaction can be understood as "specific, topic-centred regularity" in the interactional process (Voigt, 1984, p. 47), usually leading to an interactional standardisation of teaching procedures ensuring that lessons run smoothly. Forms of argumentation are characterised as specific argumentative stabilisations/interactional patterns that promote learning, allowing pupils to gain increasing autonomy in steps of action within the argumentation form (Krummheuer, 1992).

For answering the second question, a general social conceptual definition of participation is taken as a foundation (Bartelheimer, 2008) and linked with theories of the learning of mathematics (Jung & Schütte, 2017). For an adequately differentiated concept of participation it is important here to examine participation as an active, multidimensional process. In relation to the school learning process, e.g. spatial, social and didactical/content-related dimensions of participation can be differentiated (Roos, 2014). The spatial dimension of participation fundamentally relates to the time learners spend in the shared classroom space, as well as to the spatial configuration in mathematics learning together in the classroom. The social dimension focuses on social relationships (to fellow pupils, teachers and other pedagogical staff) which emerge in mathematics teaching and which mediate a participation in content-related negotiation processes. The third dimension addresses participation in didactical/content-related negotiation. Didactical inclusion relates to pupils' participation in subject teaching, focusing on their engagement with the teaching approach and content, as well as any explanations or material supplied by teachers to support the learning process.

For the purposes of analysis of didactical/content-related participation, the approaches developed in mathematics education for determining participation in collective argumentations (Krummheuer & Brandt, 2001) was used. With reference to Goffmann (1981), Krummheuer and Brandt (2001) distinguish two types of involvement in a lesson: the active, verbally productive act, and the passive, receptive non-verbal act. The aim is to identify the type of authenticity, originality and responsibility of speakers, and to identify for recipients the type of non-active participation. Learning situations become beneficial for learning according to Krummheuer and Brandt (2001) when children participate increasingly in ways that permit a shifting from minor responsibility for content and form towards greater responsibility. In this way, participation in a collective argumentation concerning statements about (mathematical) content, terms and/or methods creates the basic conditions for learning opportunities. Since this model remains more of a formal analysis of the content-related negotiation in the conversation, it is complemented by the curricular concept of mathematical activities, developed by Bishop (1988), in order to approach also the mathematical content of the activity children participate in. Bishop (1988) differentiates six activities—counting, locating, measuring, designing, playing, explaining—that are used for the analysis of moments of subject-specific mathematical participation, following Brandt (2017) and Johannson (2015). In addition, Bartelheimer (2008) focuses on the principal dynamic, i.e. the changeability of participation over time, and the impossibility of a dichotomous categorisation of inside and outside, participation and non-participation. These considerations are in tune with an interactive understanding of mathematics learning and are taken into account within the research project by the theories and methodologies applied.

5 Presentation and Analysis of a Selected Scene

The following analysis of an extract from the classroom conversation in one class provides an example of analysis using the methodological and procedural considerations elaborated above. A description of the situation will first be presented, followed by extracts from the transcript and a summary interactional analysis. The research questions will then be addressed using the presented sequence.

The selected extract comes from the classroom conversation in a Year 1 class (18th week in school). The class contains 21 children, 11 female and 10 male pupils between the ages of six and eight. Two female and two male pupils with special educational needs in the area of cognition and learning—according to the SEND Code of Practice (Department for Education & Department for Heath, 2015, pp. 97–98) with severe learning difficulties (SLD)—are inclusively schooled in this classroom. Three of these children rarely participate in whole-class mathematics teaching, as they are primarily engaged with pre-number tasks. The fourth child has a visual impairment and developmental delay. This child can partially follow the content of the mathematics teaching and can complete adapted learning tasks on the topics. Two of the children with special educational needs each have an integration assistant; an education social worker and a volunteer student also provide support in the class.

5.1 Description of the Situation

The lesson begins with set morning rituals, all of the children sitting in a circle. After the children have returned to their places, the teacher reads out a story with integrated physical exercises and does some loosening-up and stretching exercises along with the children. The children, the teacher and the support staff (Sabine and Niklas) then sit down in the circle again. At the centre there are six cloth bags in different colours, each filled with animal toys—one kind of animal in each bag. In the course of the consequent group conversation the bags are unpacked, and addition tasks are created using the respective animals from each pair of bags. This activity is embedded by the teacher in a narrative about an annual animal party. After six bags have been unpacked and three addition problems created, the teacher explains what she calls a “maths trick” and the concept of exchange tasks is introduced. In total the conversation lasts around 15 min. Afterwards, the pupils go back to their places and work on their own on exercises in the mathematics textbook.

5.2 Selected Transcript Extracts

The teacher starts the classroom conversation by asking the children whether they remember today’s date, which has already been mentioned during the morning ritual. After the date has been stated, the teacher tells a story about an animal party which takes place on this day every year, where all the animals in the world meet up on a big meadow in the woods.

1	T:	and if Moritz unpacks this bag [*gives Moritz the bag*] we can see which animals have arrived this year on the sev … on the nineteenth of January **first** to the party\
2	Moritz:	[*takes a lion from the bag and places it in front of him*] hm not a lion he lives in Africa
3	T:	[*places the lion in the centre on the carpet*] I said all animals in the world have come together and that means the lions too\ so how many lions have arrived/
4	Moritz:	[*takes another lion out of the bag and places it next to the lion on the carpet*] two
5	T:	two lions\ are there no more in your bag/
6	Moritz:	mhmh (in the negative)
7	T:	no there are no more in there\ two lions have arrived . they were the first . they have arrived here on the big meadow there are still no animals there . but perhaps Manuel can unpack the other green bag and see what animals have come next\ [*gives Sabine (Manuel’s integration assistant) the bag*]
8	Sabine:	come here Manuel see what’s in here\ [*holds the open bag towards Manuel*] see what’s in here Manuel\ here what’s in the bag/
9	Manuel:	[*takes a giraffe out of the bag and gives it to Sabine*]
10	Sabine:	oh . a giraffe . Moritz (can you put it there or so) [*gives Moritz the giraffe*]
11	Moritz:	[*places giraffe on the carpet to the right of the lions*]
12	Gabriel:	lions eat giraffes
12	T:	a giraffe has come [*takes giraffe and places it to the left of the lions*]
13	Manuel:	[*takes another giraffe out of the bag and gives it to Sabine*]
14	Sabine:	two giraffes [*gives Moritz the second giraffe*]

15	Moritz:	[*gives the teacher the second giraffe*]
16	T:	two giraffes have come [*takes the second giraffe and places it to the left of the first giraffe*]
17	Gabriel:	lions eat giraffes
18	Manuel:	[*takes another giraffe out of the bag and gives it to Sabine*]
19	Sabine:	three giraffes [*gives Moritz the third giraffe*]
20	Moritz:	[*gives the teacher the third giraffe*]
21	T:	three giraffes have come too\ … the animals have now arrived at the party\ and because this is maths/ and not science where we want to learn things about animals but we want to do some maths problems I want to ask if I made a maths problem out of it what would it be\ .. I think lots of children know … Frida*/
22	Frida:	two plus three
23	T:	the maths problem Frida is right . so I would write two for the two lions plus three giraffes have arrived [*writes the term 2+3 on a piece of paper*]
24	Moritz:	five
25	T:	hands up please .. Moritz
26	Moritz:	five
27	T:	but Nadine er er always puts her hand up quietly like a good girl you can say it too really loud for everyone
28	Nadine:	five [*holding her hand in front of her mouth*]
29	T:	look [*holds her hand in front of her mouth and says something*] (inaudible) no one heard/
30	Nadine:	five
31	T:	five animals have now arrived [*writes "=5" on the piece of paper and holds it up*] five animals have now arrived and if I wrote it as a maths problem I would write it like this\
32	Gabriel:	or also the other way round
33	T:	or the other way round\ and they make a group together\ yes/ that's a maths problem and these three and now the five of them together have lots to tell each other because they haven't seen each other for a whole year\ … now let's see who arrives to our animal party on this side\

A pupil unpacks the third bag, which contains three tigers, and under the teacher's direction places these on a new spot on the carpet. He says how many of them there are.

50	T:	three and Emilio is going to see how many come out of the same out of the are in the pink box [*points to the pink bag*] I think they'll join the tigers
51	Emilio:	that one/ [*points to the pink bag*]
52	T:	that's right have a look
53	Niklas:	yes\

54	Emilio:	[*takes the pink bag and opens it*]
55	T:	I'm excited who's in there\
56	Bastian:	iih probably a girl
57	T:	so which animals have you brought Emilio to the animal party
58	Gabriel:	gorilla
59	Emilio:	ui . (inaudible)
60	T:	a gorilla/ .
61	Emilio:	yes
62	T:	what is there still in your bag Emilio
63	Emilio:	(a gorilla and) another one [*places the gorilla in front of him and searches in the bag*] another gorilla
64	T:	yes that's a different one but they belong to the family of apes\ the apes have come too and first they say hello to the tigers\ can you put them next to the tigers/
65	Emilio:	[*puts the second ape in front of him and searches in the pink bag*]
66	Niklas:	he's looking again if there's still something in there
67	Mira:	[*tries to take hold of the pink bag and to help Emilio*]
68	Emilio:	all revealed
69	T:	are they all there all the apes or is there still another ape to come/
70	S_m:	no another one . no three
71	T:	is there still another ape to come put it here [*places the tigers a bit closer together, whereby one of the giraffes falls over*] he's been put here so nicely Bastian/
72	Emilio:	a **small** one … [*holds the third ape model in his hand*]
73	T:	oh a giraffe has already got tired [*sets the giraffe back upright*] so\ can you put them by the tigers they've now also arrived\
74	Emilio:	gorilla daddy, gorilla mummy and gorilla child [*holds the apes in his hand and places them by the tigers on the carpet*]
75	Bastian:	here [*points to the space to the left of the tigers*]
76	Moritz:	three plus three
77	Emilio:	by the tigers/
78	T:	hands up would be really great Moritz . but Mira has her hand up so what is the maths problem for the tigers and the apes\
79	Mira:	three plus three
80	T:	three plus three\ [*writes the sum 3+3 on a piece of paper*] and how many animals are there together have arrived together in this group/ Moritz*
81	Moritz:	six\
82	Bastian:	Emilio is doing it wrong\
83	T:	there are six animals in the group [*writes the result 6 on the piece of paper and puts it by the group of animals directly in front of Emilio*]
84	Niklas:	he's trying it the best he can \
85	Bastian:	ach he's stupid\
86	T:	super\ thank you Emilio

87	Bastian:	go on Emilio
88	Emilio:	three plus three gives six [*looks at the piece of paper in front of him*]
89	T:	okay
90	Emilio:	and the big fat gorilla [*examines the big gorilla in his hand*]
91	T:	and Nele is seeing which animals are still coming to our animal party [*gives Nele the dark green bag*]

Two further bags are unpacked to reveal four dogs and three pigs, which are placed together in a group on the carpet.

110	T:	here three pigs are also coming to our nice . animal party . so what's the maths problem for this . for this group of animals\ [*points to the pigs and dogs*] Thomas*
111	Thomas:	seven
112	T:	that's the result but the maths problem what is it Gabriel
113	Gabriel:	three plus four
114	T:	yes that's ri but who was there first .. Max*
115	Max:	four
116	T:	four dogs were there first [*writes 4 on a piece of paper*] and then how many pigs have come too .. er Markus*/
117	Markus:	three
118	T:	three pigs have come too [*writes +3 on the piece of paper*] and how many are there now altogether. how many animals are there now in this group altogether / [*points to the pigs and dogs*] . Nele*
119	Nele:	seven
120	T:	exactly seven [*writes =7 on the piece of paper*]

5.3 Summary Interactional Analysis

Lines 1–33

After the teacher has started the classroom conversation with the story about the annual animal party on the meadow, she involves a pupil, Moritz, actively in the story, asking him to unpack animal toys from one of the bags <1>. By emphasising the word "first" the teacher highlights the importance of the sequential nature of the procedures in the story she has set up, and in addition establishes the suspense typical of stories. At this moment, the question in the focus of the interaction is which animals are going to come to the announced party. Moritz follows the teacher's request and unpacks an animal toy from one of the bags; he immediately determines the kind of animal: "hm not a lion he lives in Africa" <2>. However, he seems to see a discrepancy between the teacher's story and the animal toy he has found in the bag. The lion he has taken out of the bag seems to him not to fit the story because it lives in Africa. The teacher addresses these doubts by repeating that all animals in the world can come to the party, which includes lions <3>. She thus acknowledges Moritz's utterance and at the same time shows that it is not for the children to decide whether or not certain animals are right for the story. She then asks after the number of lions that have arrived to the party, thus shifting the interest from the kind of animal onto the enumeration of certain groups of objects—here the animal toys <3>. Thus, to an extent Moritz's task changes too, from naming the kind of animal to enumerating the animals of a certain kind. Moritz engages in the task by getting all the animal toys out of the bag and saying "two", meaning there are two lions <4>.

The teacher then asks again whether the bag is really empty <5>. This can be interpreted on the one hand as a real question, asked because she does not know how many animals are in the bag; but it could also be interpreted as an interactional step that makes the procedure explicit once again for all the children: the bags with the animal toys are to be opened and all the contents unpacked. After Moritz confirms that there are no more lions in the bag <6>, the teacher ends Moritz's active (acting) participation in the story by once more summarising the current situation: "no there are no more in there\ two lions have arrived. they were the first. they have arrived here on the big meadow there are still no animals there" <7>.

The teacher now actively engages another child, Manuel, in the telling of the story, allowing him also to unpack animals from one of the bags. Manuel is supported in this by his integration assistant, who involves another child, again Moritz, in the action-based activity of unpacking the animals and placing them on the carpet <8–10>. Thus, all the giraffes are unpacked one by one from the second bag. At this point, another pupil, Gabriel, again questions the story on the basis of the kind of animal <12>. However, the teacher does not address this utterance. It becomes clear that the children can and should contribute to the story within the framework of the activities set out by the teacher, but that they cannot participate in shaping its content.

After all animals have been unpacked from the first two bags, the teacher indicates the context of mathematics teaching, explicitly setting a shift in focus from the animal toys and their characteristics onto mathematical tasks that are represented by the animals. The teacher asks about a maths problem: "I want to ask if I made a maths problem out of it what would it be" <21>. The problem is formed by Frida—"two plus three" <22> and then solved by Moritz and Nadine. The maths problem represented by the groups of animals is noted in written form. Gabriel, who was previously distracted by the story and indirectly questioned the coming together of lions and giraffes <12, 17>, has switched his focus, following the teacher, from animal story to mathematics, and contributes to the interaction by saying that the maths problem could also have been written the other way round <32>. This utterance is imitated by the teacher but not further addressed, such that for many of the children it presumably remains unclear what the utterance "or the other way round" relates to. In the sequence examined here, a pattern is initiated for the interactional procedure, which is consequently repeated twice: unpacking a bag, determining the kind of animal, enumerating the respective toys, and formulating the corresponding addition problem, which is then written down by the teacher.

Lines 50–91

In the second repetition of the interactional procedure, Emilio unpacks the second bag. Again the kind of animal is determined <57, 58>. However, Emilio also allocates family roles to the animals and takes a long time arranging his animal toys alongside the others on the carpet <74>. Meanwhile, in the classroom conversation, the corresponding addition problem is formulated, solved, and written down on a piece of paper by the teacher <79, 80>. The teacher places the piece of paper with the problem in front of Emilio, who then reads it out loud <88>. Emilio's subsequent utterance "and the big fat gorilla" <90> indicates however that his thoughts are still preoccupied principally with the animal toys and their relationships to each other—in this case based on their physical characteristics.

Lines 110–120

In the third repetition of the interactional procedure both bags have already been unpacked; the types of animal have already been determined and the respective animals enumerated. The teacher asks after the maths problem as follows: "here three pigs are also coming to our nice . animal party . so what's the maths problem for this . for this group of animals" <110>. Thomas answers by giving the result of the maths problem <111>, thus seeming to show awareness that the teacher's individual questions point not to the maths problem itself but to its solution. He thus abbreviates the interactional procedure that has been established up to this point to go straight to the presumed heart of the matter, the mathematical solution. The teacher does not permit this abbreviation and again asks specifically for the maths problem: "that's the result but the problem what is it" <112>. Gabriel, who at the beginning of the classroom conversation had already pointed out that each formulated addition problem could also be written in reverse order, responds to this by completing the interactional step of naming the problem, but without considering the sequential ordering of the story <113>. Here, too, the teacher intervenes and narrows down the children's possible answers by asking for specific enumerations, herself connecting these to the context of the maths problem corresponding to the interactional procedure <116, 118>.

5.4 Answering the Research Questions

What patterns and structures can be reconstructed in the classroom conversations?

The overall classroom conversation can be divided into repeated sequences. One such complete sequence can be found in <1–33>. The structure described here is also shown in the other two sequences:

1. First bag is unpacked
2. Kind of animal in the first bag is determined

3. Number of animals in the first bag is determined
4. Repetition of the steps 1–3 with another bag
5. Calculation is formulated
6. Result of the calculation is determined
7. Calculation is noted in written form

This structuring of the classroom conversation into repeated sequences is reflected also in the other analysed classroom conversations. Within these sequences, a progression can usually be observed from action- and material-oriented interactional steps—sometimes without any direct relation to mathematical contents/activities—towards formal interactional steps with a relation to mathematical content.

Which opportunities for participation are created for different children (within these patterns and structures)?

Considering the participation of different children in the interaction in relation to its structure, it is first of all noticeable that the opportunity to participate is maintained for all pupils in the class through the different demands posed in the individual interactional steps, and several different children take part in the classroom conversation. Considering the participation of the two boys Emilio (child with SEN) and Gabriel (child without SEN), in relation to the spatial dimension of the classroom conversation, the following can be observed. During the classroom conversation both Emilio and Gabriel are seated in the circle with the other pupils. They have both chosen their place in the circle themselves. After Emilio chooses his place the volunteer student sits down next to him on his left; on his right is a female pupil. Gabriel sits between two female pupils. Considering the social dimension, it is noticeable that both the boys participate actively in the classroom conversation and put up their hands to show they want to take part. During the classroom conversation Emilio carries on short conversations with the volunteer student, who keeps trying to get him to be quiet and pay attention. Emilio reacts to some of the other pupils' utterances that are not related to the classroom conversation with laughter and sounds of agreement. Gabriel briefly communicates twice with his neighbours during the classroom conversation.

Considering the boys' participation in the didactical/content-related dimension, it is noticeable that Emilio participates above all in the first steps of the described procedure of the sequence <51, 54, 59>; thus, the action- and material-oriented steps are the focus for him. Utterances where Emilio is responsible for content, formulation and vocalisation <72, 74, 90> relate above all to the characteristics of the animal toys and their relationships to each other. According to Bishop's categories, his activities can be categorised principally under measuring. In later steps of the described sequential procedure Emilio participates as an imitator, reading out the problem that has already been formulated and written down <88>. Gabriel, on the other hand, takes part in later steps of the described procedure <32, 113>. In his utterances Gabriel takes responsibility for content, formulation and vocalisation, and can be characterised as a creator. His activities can be categorised under counting, although it is noticeable that, released from the concrete task, he points out structural aspects of addition problems <32>.

6 Summary Results and Perspectives

The results of the first analyses show that the group classroom conversations were often structured in similar short sequences with four to five repetitions. These individual sequences show a particular, small-steps path from action- and material-oriented to formal interactional contributions. Thus patterns of interaction emerge to establish collective argumentation. These repeating patterns allow the participation of many children with different abilities. A more precise analysis of the opportunities for social and content-related participation of various children within these classroom conversations on the one hand shows clear differences in terms of the individuals involved in social interactions (fellow pupils, teacher, pedagogical staff). On the other hand, in terms of content-related participation, the lower-achieving children participate autonomously above all in early action- and material-oriented steps of the pattern of interactions; a participation in later, more formal steps takes place above all through imitation of contributions already negotiated in the interaction, with reference back to the action- and material-oriented steps. The example shows that the mathematical contents of the activities children participate in can differ even if the children participate in the same classroom discussion.

High-achieving children participate in all steps of the example interactions. However, it can also be seen that in many cases their contributions "abbreviate" the pattern of the interaction and put the procedure into question, although they are always guided back to the example. These kinds of patterns emerging in inclusive teaching parallel the results of analyses of regular classes and appear to be further reinforced when there are even greater differences in the learners' levels of achievement. Finally, it should be noted that the goal of inclusion cannot be for all learners to participate in a classroom conversation in the same way. Rather, learners should receive learning opportunities that are appropriate for their abilities and potentials for content-related participation. Under the current conditions of teaching, it remains to be negotiated in detail what such a compromise might look like to ensure a group exchange providing conditions that promote learning for all participants.

Notes—Transcription Notation

Bold	Spoken with emphasis
Smaller	Spoken quietly, whisper
[action]	Action that takes place between two temporally separate sections of the transcript
(word)	Word not clear/incomprehensible
/	Voice inflected up
\	Voice inflected down
. .. …	Pauses in speech, in seconds
*	Student has raised his/her hand

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A Teacher's Use of Revoicing in Mathematical Discussions

Kaouthar Boukafri, Marta Civil and Núria Planas

Abstract This study explores how a teacher's use of revoicing promotes students' mathematical thinking and, more generally, mathematical learning opportunities. We analyzed four lessons where 12-year-old students solved geometry problems. We identified episodes that illustrate how the teacher's actions supported students' explanations during mathematical discussions. In this chapter we show two examples, from one of the lessons, where the teacher's use of revoicing created spaces where students strengthened their understanding of the concept of distance between two points and its relation with the Pythagorean Theorem. Our theoretical approach to revoicing leads us to distinguish and examine three dimensions: linguistic, discursive and mathematical. The integrated view of such dimensions serves to find emergent relationships among talk, classroom discourse, and learning opportunities.

1 Introduction

The larger study on which this chapter is based focuses on the use of whole class discussion as a way to promote students' understanding of mathematics. The cited study is part of the Ph.D. research of the first author. Interaction in the course of group discussion provides opportunities for students to learn mathematics (i.e., mathematical learning opportunities) for instance, as they compare their approach with others' or when they try to justify their own ideas (Yackel & Cobb, 1996).

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© Springer International Publishing AG, part of Springer Nature 2018
J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_12

Moreover, the fact that students share their ideas and thoughts in the context of a full class discussion helps them to both organize and clarify their own reasonings (Pimm, 1987). We build on Sfard's (1998) participation metaphor, in which the teacher's role is that of expert participant who encourages students to refine the ideas they shared so that they are ultimately able to formulate mathematical explanations.

Research on the use of discussion in the mathematics classroom points to a series of moves that teachers can make to promote students' understanding of mathematics (Stein, Engle, Smith, & Hughes, 2008). One such move would be to orchestrate the broad range of students' answers to a task so that they deepen in their understanding of the mathematical content involved. The teacher has to be able to guide the discussion to take advantage of all the interventions as they work, as a class, toward solving the mathematical task. Another teacher's move in classroom discussions is what has been called *revoicing* (e.g., Empson, 2003; Enyedy et al., 2008; Forman & Ansell, 2001; Moschkovich, 2015; O'Connor & Michaels, 1996). Quoting Enyedy et al., "Revoicing occurs when one person re-utters another's contribution through the use of repetition, expansion, or rephrasing" (p. 135). According to these authors, the relevance of revoicing does not simply lie in how the teacher re-utters the students' contributions but in what this re-utterance allows to happen concerning their role within the whole class: it positions students as authors of ideas and involves them as developers of the discussion. Furthermore, it allows the teacher to guide all interventions toward a more formal mathematical formulation. In this chapter we focus on the analysis of the use of revoicing by a teacher to guide the discussion in order to let the students construct mathematical processes as well as concepts. Our understanding of revoicing as an object of investigation itself is summarized in the next section.

2 Our Approach to Revoicing

We approach revoicing through three dimensions, namely, linguistic, discursive, and mathematical; and from here we draw on three related notions: linguistic form, discursive function, and mathematical effect. The first two dimensions come from our sociolinguistic orientation to research in mathematics education. The third dimension allows us to select linguistic and discursive issues in relation to moments of mathematics teaching and learning. The study of teacher revoicing is aimed at identifying spaces in the classroom discussion where students can strengthen their learning of mathematics.

By *linguistic form* we mean the literal occurrence of revoicing in the talk. In their studies Forman and Ansell (2001, 2002) observed that teachers have a tendency to repeat, expand, recast, translate, rephrase, and report students' mathematical explanations. In this chapter, we focus on four of these tendencies as non-exclusive types: repeating as literally resaying, expanding as adding new content, rephrasing as reshaping the same content, and reporting as recalling in a summarized way. By *discursive function* we mean the impact of revoicing in the immediate subsequent turns. Here, we can find types which again are non-exclusive: clarifying the content, introducing new ideas, explaining the reasoning more precisely, reorienting the discussion (O'Connor & Michaels, 1996), and reformulating terms (Moschkovich, 2015). These examples served as inspiration in the search of emergent discursive types from our data. Finally, by *mathematical effect* we mean the construction or development of mathematical content during the interaction. The mathematical types also emerge from data. For example, we have found the usage of concatenated segments to construct a full path, the slope of a straight path, or the identification of the ends of a segment. This gives rise to the construction of the concept of distance between two points in the plane (see our findings below).

In this research, to analyze the use of revoicing within the context of a whole class discussion, we have selected turns where we detected revoicing with explicit mathematical activity, meaning that they contained explicit allusion to mathematical content. Hence, we have not considered any turns with revoicing that concerned orchestration of the whole class activity. This implies that we had to search for expressions with mathematical content in speech turns, that were related either to mathematics in a general way or to the activity the students were involved in at that moment.

3 Participants, Task and Methods

The students were 12-year-old secondary school students from Barcelona, Catalonia-Spain. The teacher had nine years of experience teaching mathematics and eight years of experience doing research in mathematics education. When teaching, the teacher promoted problem solving and encouraged students to share strategies. The students would work in small groups (often in pairs), participate in whole class discussions moderated by the teacher, and finally refine their individual write-ups.

For this study the students worked on geometry problems during four lessons. According to Smith and Stein (2012) the selected mathematical activities were pertinently adapted mathematical problems that dealt with contents of the current curriculum and that could be solved via a number of different strategies (Schoenfeld, 1985).

We focus on the problem of 'The Spider and the Fly'[1] (see Fig. 1). The goal is to find the shortest path for the spider to catch the fly. The students worked in pairs as they engaged with a three-dimensional scale model of the room and used stickers to simulate the spider and the fly. They were encouraged to explore the problem using the different materials given.

[1]Adapted from http://nrich.maths.org/2365.

The Spider and the Fly
A spider is sitting in the middle of one of the smallest walls in my living room and a fly is resting by the side of the window on the opposite wall, 1.5m above the ground and 0.5m from the adjacent wall (the one that has pictures hanging on the wall). What is the shortest path the spider would have to crawl to catch the fly?
The room is 5m long, 4m wide and 2.5m high.

Fig. 1 Translation from Catalan of the problem

All whole class discussions were videotaped and transcribed literally, following the chronological order of interventions of each participant in the discussion. Each turn corresponds to an intervention done by one single participant, regardless of the time spent on it or the information he or she talked about.

The first author read the transcripts and grouped the turns that are constitutive of reasoning related to a particular mathematical content, both conceptually (e.g., the distance between two points in the plane) and procedurally (e.g., the application of the Pythagorean theorem). She identified specific terminology (Pimm, 1987) related to mathematical content, including that of the context of the problem under consideration. The mathematical types aid the identification of relations between different mathematical content. Such groups are named episodes in our work. In each episode, we looked for turns where revoicing was identifiable in the teacher's responses to students. First, revoicing was identified according to its linguistic form. After having found the linguistic types for a number of turns, we examined the discursive function for each of them. Once we had determined both the linguistic and the discursive types, and in order to identify mathematical effects, we examined the entire episode from the perspective of what happened mathematically. In this chapter we have numbered the identified revoicing (that appear in the transcription labeled as <R_i>) to make it easier to refer to them at any moment.

Following the framework of multiple voices (Barwell, 2012), it cannot be expected that the impact of any revoicing in the immediate subsequent turns is univocally determined. In the team, we elucidated and then discussed different explanatory alternatives for the impact of each revoicing, that is, its discursive function. From here, we decided the most feasible alternative in terms of which has a clearer presence of the particular mathematical content in the episode. Thus, the discursive type is used as an analytical bridge between the linguistic and the mathematical types. This is in line with our approach to mathematical learning opportunities grounded on the sociolinguistic and discursive domains. In the section of analysis below, we directly provide the chosen discursive function (e.g., second column of Tables 1 and 2) and leave the discussion of all alternatives for another publication.

Following the framework of learning as participation and communication in discourse (Sfard, 2001), we take the construction and development of mathematical processes and concepts during the interaction as evidence of mathematical learning opportunities. The mathematical effect is not intended to claim whether the students have come to learn (i.e., reconstruct, develop) either mathematical processes or concepts. In the team, drawing on the discursive functions, the transcript of the episode and the video, we elucidated, when possible, and then discussed mathematically significant turns in the talk (e.g., first column of Tables 1 and 2). From here, we analyzed opportunities to learn mathematics by means of at least a narrative representation. A diagrammatic representation (third column of Tables 1 and 2) was added in the analysis of episodes in which some of the participants used graphical resources to support their explanations. Finally, we searched potential relationships among the identified mathematical effects and the emerging opportunities from them.

4 Analysis of Two Episodes

'The Spider and the Fly' deals with relationships between two and three dimensions, properties of polyhedra, the Pythagorean Theorem, and the distance between two points. We have selected two out of the five episodes that constitute the whole group discussion. The first selected episode for this chapter (Sect. 4.1, Episode 1) deals with the distance between two points in the plane. The second selected episode (Sect. 4.2, Episode 2) deals with the Pythagorean Theorem. In the following we are going to describe each of these two episodes: we will give details on the transcripts that contain interventions of the participants with identified revoicing, and give details of the analysis concerning the three dimensions we have explained above—linguistic, discursive and mathematical.

4.1 Episode 1: Distance Between Two Points

In the first episode, the notion of distance between two points comes to be collectively constructed. Sara[2] and Cris, two of the students, have each presented their solution method to the class. Sara's group suggests the optimal path to be three concatenated segments, each of them parallel to an edge of the room. The teacher tells them that they need an extra segment to reach the fly. At this point, the teacher begins by taking one of the scaled two-dimension models, and draws Sara's and Cris' approaches on the board (Fig. 2a, b).[3] Then she makes the whole class focus on these two approaches through revoicing (R_1 and R_2). She revoices again at the end of the episode (R_3). This is a translation from Catalan and Spanish of a partial transcript of the episode:

2 Sara:

[2]All names are pseudonyms.

[3]In Fig. 2, the piece labeled "Wall Pictures" refers to a wall that had pictures hanging on the wall. The students' three-dimensional models showed that.

We wanted to go up for one point twenty-five; after that, five metres straight and finally one to reach the fly

(…)

8 Cris: We've thought about going two metres straight, then to follow the diagonal of the side, and then…

9 Teacher: **<R$_1$>** Let's do it graphically, what Sara said: the spider goes all the way up to this point [Ai, Fig. 2a], and then goes here [Af, Fig. 2a]. Does everyone see that these two walls are connected? It appears here [ceiling], and walks five metres. It appears here [window wall] and goes down one meter. After that, it still has to come here [fly]. It's a staircase path.

<R$_2$> They [Cris' group] suggest that the spider goes in straight line in this zone and also here [fly wall]. And instead of doing a staircase path they follow in diagonal [she draws a line through the pictures wall]. And then they did a diagonal here [pictures wall]

10 Maria: But the best solution is to do a diagonal from the spider to the fly.

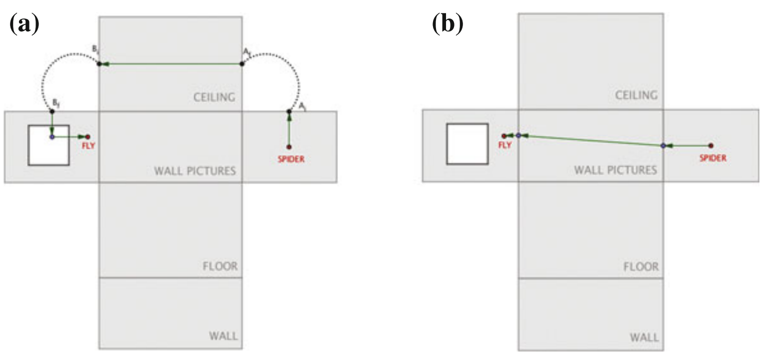
11 Teacher: Only a single straight line? Then, from this point [fly] to this other point [spider] directly with a diagonal. Thus, without doing any straight segment [in her drawing she adds a straight line from the spider to the fly] Will it be a shorter path?

12 Students: Yes.

13 Teacher: Okay. This [drawing] is the best option of these two proposals. How do we know that the diagonal path is shorter than doing two straight segments?

14 Albert: Because you follow a straight line.

15 Teacher: **<R$_3$>** Always the shortest distance is given by a straight line.



Sara's group approach

Cris' group approach

Fig. 2 **a**, **b** Representations of two approaches to the shortest path

In R_1, Fig revoices the suggestion by Sara that the solution path has to be a polygonal line. It is a revoicing that includes both Sara's initial proposal of a three-sided polygonal line and the correction: the need of addition of a fourth segment to reach the fly ("After that it still has to come here"). In R_1 we see two linguistic types: reporting and expanding. On the other hand, the discursive function of R_1 is to outline an inadequate understanding of the situation in the wording.

In R_2, concerning Cris' approach, Fig revoices the suggestion of a three-sided polygonal as solution and emphasizes twice the fact that the spider follows a "diagonal path".[4] After R_1 and R_2, Maria proposes following an oblique path. At this point Fig focuses on Maria's and Cris' proposals, leaving Sara's aside; she affirms Maria's to be the best option, though she keeps asking students for justification. The teacher draws this third option and asks to validate whether Maria's proposal is so far the shortest path. Albert justifies the best path to be the one formed by a straight line; he gives a general statement ("a straight line") without referring to the property of inclination of the line. Thus, the linguistic type of R_2 is reporting, and its discursive type is emphasizing an idea (i.e., the use of diagonal segments) by comparison and repetition.

In R_3, the teacher uses Albert's "straight line" in combination with the goal of the problem (i.e., finding the shortest path) to say, "Always the shortest distance[5] is given by a straight line". After R_1 and R_2 Maria and Albert build on their peers' solutions to progress toward the idea that a straight line is the shortest path between two points, though it is the teacher who explicitly refers to the concept of distance in R_3. Hence, the linguistic types of R_3 are expanding and repeating, and the discursive type is contributing to the formalization of a concept (i.e., the concept of distance) by definition.

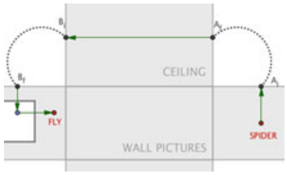
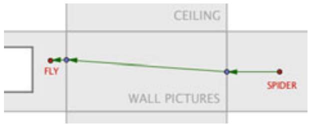
The teacher focuses on Sara's and Cris' ideas and, through revoicing (R_1 and R_2), she encourages students to improve and reformulate their peers' approaches. It is interesting to note that she does not conclude right after Maria's intervention, but allows for the exchange to go on a little longer and revoicing comes after Albert's intervention (R_3).

Taken together, R_1, R_2, and R_3 have the accumulative mathematical effect of building the concept of distance between two points in the plane. In R_3 the teacher formalizes the idea that the distance between two points equates the length of the segment joining them. The third column of Table 1 illustrates the evolution, within the episode, of how the concept is discussed and defined. R_1 contributes to the discussion about polygonal paths; R_2 leads to change the proposal by considering oblique segments; and R_3 leads to the segment solution. These mathematical effects point to the creation of spaces in the classroom discussion where students can strengthen their learning of mathematics and, therefore, to the creation of

[4]The term "diagonal" is widely used in an informal way to refer to oblique segments in school mathematics.

[5]In the usage of the word path (see Fig. 1) instead of the word distance, the teacher still explicitly refers to the property of being the shortest that is, in turn, the definition of distance.

Table 1 Three-sided typifying of revoicing—Episode 1

	Linguistic form		Discursive function	Mathematical effect
R_1	*Let's do it graphically, what* ***Sara said****: the spider goes all the way up to this point [A_i], and then goes here [A_f].*	Reporting	Outlining an inadequate understanding of the situation in the wording	
	After that, it still has to come here *[fly].*	Expanding		
R_2	***They*** *[Cris' group]* ***suggest that*** *the spider goes in straight line in this zone and also here.*	Reporting	Emphasizing an idea by comparison and repetition	
R_3	***Always the shortest distance is given by***	Expanding	Formalizing a concept by definition	
	a straight line	Repeating		

mathematical learning opportunities. In particular, when revoicing the teacher excludes those parts of the students' reasoning which do not solve the problem (e.g., polygonal paths). In this way revoicing acts as a filter for student contributions.

4.2 Episode 2: The Pythagorean Theorem

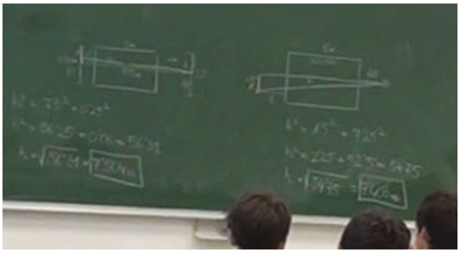
The second episode is a discussion focused on providing support for the argument that the answer from Episode 1 was correct by computing the path length. To do so, the teacher encourages the students to start finding the length of the path that goes through the wall with the square-pattern on it (patterned wall from now on). The teacher starts by asking for the length of this wall and right after that the students (Albert, Carles and Jana) start dictating all the required data. In this episode, all identified revoicing deals with the data that these students dictated (namely R_4, R_5, R_6, R_8, R_9, and R_{10}) and also with the hypothesis that the path that is the solution is actually an edge of a right triangle (R_7). During the lesson the teacher writes down the data provided by the students on the blackboard (Fig. 3). This is a translation from Catalan and Spanish of a partial transcript of the episode:

46	Albert:	Five.
47	Teacher:	**<R$_4$>** Five metres. And the spider is here.
48	Albert:	It was right in the middle.
49	Teacher:	How much was that?
50	Charles:	Two on the pictures wall.
51	Teacher:	**<R$_5$>** Two metres. And here we had the fly, that was at…?
52	Albert:	Zero point five.
53	Teacher:	**<R$_6$>** At zero point five. Okay, if I draw this diagonal, do you all remember how to compute its length? Okay, what I get here is a diagonal of what?
54	Jana:	Of a right triangle.
55	Teacher:	**<R$_7$>** Of a right triangle. Of a right triangle like this. Where this height, how long is it, exactly?
56	Students:	Zero point twenty-five.
57	Teacher:	**<R$_8$>** Zero point twenty-five. Did everyone get that?
58	Students:	Yes.
59	Teacher:	And what about its base?
60	Albert:	Seven point five.
61	Teacher:	**<R$_9$>** Seven point five. Where did you get this seven point five from?
62	Students:	It's five plus two…
63	Teacher:	**<R$_{10}$>** The five plus two plus zero point five. If we all do this now, you… this diagonal is the so-called hypotenuse, isn't it? Hence you want to get its length using the Pythagorean Theorem. […]

(…)

67	Teacher:	[…] Then, let's focus on the two candidate options we have now. We got that it will be seven point five, seven point five zero four if we go through the patterned wall or seven point four three if we go through the ceiling. Which option should we choose?

Fig. 3 Application of the Pythagorean Theorem for the two paths



Most of the identified revoicing in this episode consists in guessing the numerical data that would lead to determine the length of a path that is a candidate to solution, in this case, going through the patterned wall. This path consists of a single straight segment that joins the position of the two animals, and that can be drawn as the hypotenuse of a right triangle. In R_4, the teacher revoices the length of the patterned wall that Albert shares, which was information in the problem statement. In R_5, the distance between the position of the spider and the patterned wall is computed. This is possible since we know that the position of the spider is halfway (between the ceiling and the floor) of the small wall in the room, as Albert notices. That is, at a two meter distance. Finally, in R_6, the distance between the fly and the patterned wall is determined, and it is zero point five metres. All together, R_4, R_5 and R_6, allow students to determine the base of a right triangle. In R_7 the teacher revoices the relationship between the diagonal segment and the right triangle with the optimal path between the spider and the fly going through the patterned wall.

In R_8 the teacher revoices the height of the right triangle and in R_9 the length of the base that is obtained as the addition of the lengths discussed in R_4, R_5 and R_6.

In R_{10} the teacher repeats how to obtain the length of the base as the so-cited addition, and explicitly relates the path with the hypotenuse of a right triangle for which both base and height are known. Hence, one can use the Pythagorean Theorem to determine the desired length.

The episode continues with the determination of the right triangle whose hypotenuse is the optimal path between the spider and the fly, but now through the ceiling, and the explicit numerical computation of the lengths of the two candidate paths via the Pythagorean Theorem. This leads to the conclusion that the best option is going through the ceiling.

It is necessary to know the length of two of the three sides of a right triangle in order to get the length of the third by means of the Pythagorean Theorem. The identified revoicing in this episode can be understood as guidance to the determination of the length of the base (R_4, R_5, $R_{6,}$ $R_{9,}$ and R_{10}) and of the height (R_8) of the right triangle (R_7) that has the optimal (shortest) path between the spider and the fly as hypotenuse. In the present case it refers to the path through the patterned wall. The teacher uses revoicing to emphasize the data that is needed to determine both lengths, and to do so the teacher literally repeats the interventions of the students. The data that the students provide come either directly as such from the statement of the problem or from making sense of that statement (or the problem?). The third column of Table 2 illustrates the contribution of each piece of data and how all students have the opportunity to follow the construction of the right triangle step by

Table 2 Three-sided typifying of revoicing—Episode 2

	Linguistic form		Discursive function	Mathematical effect
R_4	***Five*** *metres*.	Repeating	Emphasizing data by identification and interpretation	
R_5	***Two*** *metres*.	Repeating		
R_6	***To zero point five***.	Repeating		
R_7	***Of a right triangle***.	Repeating	Emphasizing an idea	
	Of a ***right triangle like this***.	Expanding		
R_8	***Zero point twenty-five***.	Repeating	Emphasizing data by interpretation	
R_9	***Seven point five***.	Repeating		
R_{10}	*The five plus two* ***plus zero point five***.	Expanding	Getting results by connecting data	

step, and how it leads to the determination of the length of the candidate path using the Pythagorean Theorem.

It is important to note that unlike what happens in Episode 1, here the teacher uses all the contributions made by students. The teacher does not exclude any of them. Here we could say that in this way revoicing acts as a builder of contributions.

5 Some Final Remarks

We have found that this teacher's revoicing in the whole group discussion provided opportunities for students to construct the concept of distance between two points and to relate the final computation of the length of the solution path as a direct application of the Pythagorean Theorem. In the first episode, students had the opportunity to both listen to a correct use of formal mathematical language (Moschkovich, 2015) and also approach the formal definition of the concept of distance between two points in the plane. The notion of distance in the episode was obtained by building on the contribution of students to the discussion. The ways that the teacher used revoicing served as a filter for those contributions, developing an approach to the solution with language that became more formal and precise. In the second episode all student interventions were used to build the right triangle whose hypotenuse is the optimal path that joins the spider and the fly.

This analysis of teacher discourse moves is confirmed as a window into the study of mathematical learning opportunities (e.g. Sfard, 1998, 2001). Revoicing mediated the process that went from the discussion of the students' initial ideas to the construction of mathematical explanations. In our data, revoicing was a teacher's practice generated by the students' participation. On the one hand, the teacher guided the discussion in ways that allowed students to take the initiative and, on the other hand, the students were ready to follow up the teacher's revoicing. The mathematically productive mediation of revoicing suggests the need for more investigation of this teacher move. This practice also deserves research attention when revoicing is carried out by students. The insights produced in understanding teachers' revoicing may be useful in understanding students' revoicing, in both whole class discussions and peer work.

Acknowledgements This investigation is developed in the context of the Programa de Doctorat en Educació de la Universitat Autònoma de Barcelona, of which the first author is a student. We thank the school, the students and the teacher. Our funding comes from Project EDU2015-65378-P and Grants FPI BES-2013-063859 and EEBB-I-15-10216, Spain, and GIPEAM 2014 SGR 972, Catalonia.

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Authority and Politeness: Complementary Analyses of Mathematics Teaching Episodes

Konstantinos Tatsis and David Wagner

Abstract In this chapter we juxtapose two analyses of episodes from a year 9 mathematics class. Particularly, we analyse the ways that a mathematics teacher managed authority relationships when he moved from a familiar context to an unfamiliar one in a much larger school. We build off an analysis using authority structures, following previous research. Then we compare that to an analysis based on politeness theory, with a focus on the effect of verbal acts on the participants' faces. Additionally, we investigate how various verbal acts affect the definition of the situation. We conclude by comparing the revelations from the two conceptual frames; we claim that politeness theory may help us explain teachers' and students' choices of particular authority structures in their classroom interactions.

1 Introduction

In the mathematics classroom, teachers are expected to have authority due to their position and their knowledge and at the same time develop their students' sense of authority within the discipline of mathematics. This tension poses challenges for mathematics teachers and students, who need to cope with a range of issues simultaneously. Teachers and students can be seen as involved in the same social situation: firstly, they are confined in the same space—the classroom and the school building. Secondly, they are expected to adhere to particular rules and norms (e.g. Yackel & Cobb, 1996), which are, or can be, subject to negotiation during the interactions. At the same time, a series of expectations concerning the 'outcomes' of

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© Springer International Publishing AG, part of Springer Nature 2018
J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_13

schooling are at play; some of these come from within the school-classroom system, while others come from external sources such as parents or state authorities, e.g., as expressed in curriculum documents. Thus, for the analysis of the interactions that take place in a mathematics classroom a combination of theoretical and methodological approaches may be helpful (e.g., Bikner-Ahsbahs & Prediger, 2014; Tatsis & Dekker, 2010). In this chapter we juxtapose two analyses of episodes from a year 9 mathematics classroom in Canada. To do this, we firstly build off an analysis using the conceptual frame developed by Herbel-Eisenmann and Wagner (2010) to identify the way obligation works in mathematics classroom relationships; then we compare that to an analysis that uses politeness theory, as proposed by Brown and Levinson (1987) and implemented in mathematics education (e.g. Rowland, 2000; Tatsis & Rowland, 2006). With this juxtaposition, we ask what the politeness theory adds to the obligation framework in terms of what phenomena it highlights and why the obligation structures may be attractive in some ways for mathematics teachers. This leads us into a reflection on possible courses of action for mathematics teachers. Moreover, we investigate the compatibility of the authority framework and politeness theory on a theoretical and methodological level.

2 Positioning and Authority

The conceptual frame developed by Herbel-Eisenmann and Wagner (2010) draws from a quantitative analysis of the most pervasive speech patterns in mathematics classroom interaction, and builds from positioning theory. Van Langenhove and Harré (1999), who are central figures in this theory, have described positioning as the ways in which people use action and speech to arrange social structures. In any interaction, the participants envision known storylines to help them interpret what is happening. These storylines may be conscious or not. They can be contested explicitly or implicitly. A powerful aspect of this theory is its radical focus on the immanent—its rejection of the transcendent. In other words, the theory considers real only that which is present in the interaction and rejects the power of exterior forces. In an analysis of the way this theory was taken up in mathematics education research, Wagner and Herbel-Eisenmann (2009) noted that the discipline of mathematics and other exterior forces may be myths, but they can be taken as real in classroom or other interactions because teachers and others may be viewed as representatives of these exterior forces. Positioning theory's focus on the immanent motivated Wagner and Herbel-Eisenmann's (2009) focus on the interactions among the people in the classroom instead of, for example, the way the people relate to or feel about mathematics. Nevertheless, it is illuminating how the personal relationships construct or reflect the discipline of mathematics.

3 Politeness Theory and the Notion of Face

The aforementioned ways in which people use verbal acts to arrange social structures have been the focus of various linguistic and social theories. Our focus on the interactional aspects of the classroom encounters leads us to adopt an approach that puts the interaction at the fore: symbolic interactionism, as introduced by Mead (1934) and Blumer (1969) and elaborated by Goffman (1971, 1972), considers the role of symbols as vital for the process of interactions. It is through verbal and non-verbal symbols that people establish shared meanings and define the situation they are involved in. The definition of the situation is common among the participants; however, it "involves not so much a real agreement as to what exists but rather a real agreement as to whose claims concerning what issues will be temporarily honoured" (Goffman, 1971, p. 21). This definition may be affected by:

(a) the history of the situation,
(b) some characteristic aspects of the participants and their relations,
(c) the contexts in which the situation is contained,
(d) the desired outcome of the situation.

The history of the situation is constructed by the participants' interpretations of events that have taken place; these storylines are crucial in what the participants expect from a situation. The participants' characteristic aspects may be external (e.g., related to one's appearance or gender) or may be based on already established relations and existing authority distributions (e.g., related to one's position). The context of the situation may influence its definition in a micro-level (a particular verbal exchange between two students), a meso-level (the mathematics classroom) and up to a macro-level (the education system or any other larger system). Finally, the desired outcome of the situation is closely related to its history and leads to the establishment of some expected behaviours by the participants. These behaviours are in turn related to some prescriptions, which the participants agree to follow, implicitly or explicitly. Prescriptions are defined as "behaviours that indicate that other behaviours should (or ought to) be engaged in" (Biddle & Thomas, 1966, p. 103) and can be further categorised to demands or norms, according to their overt or covert nature, respectively. Teachers and students are bound by a set of demands and norms that refer to a wide range of behaviours and routines, ranging from the time and duration of schooling to the conventions used to address each other. While, as we mentioned, demands are overt, thus more stable and easier to observe, norms are harder to observe: Homans (1966) described them as ideas in the minds of people, ideas which "can be put in the form of a statement specifying what the members [of a group] or other men should do, ought to do, are expected to do, under given circumstances" (p. 134). These ideas, in the form of understandings or interpretations, become normative or taken as shared by the participants in a given situation (Yackel, 2001). A number of researchers have investigated the influence of norms in the establishment of mathematical meanings in a classroom community. One of the basic premises of such analyses is the differentiation between social and sociomathematical norms: the former refer to behaviours related to any situation, while the latter refer to “normative aspects of mathematics discussions specific to students’ mathematical activity” (Yackel & Cobb, 1996, p. 461). A characteristic social norm that has been identified during collaborative problem solving among students (but is also valid for most social interactions) is the ‘avoidance of threat’ norm (Tatsis & Koleza, 2008): the speaker is expected to use such verbal acts in order to avoid any potential threat to the other’s face. Face has been defined as “the positive social value a person effectively claims for himself [sic] by the line others assume he has taken during a particular contact” (Goffman, 1972, p. 5) and it is the central concept in politeness theory. Politeness theory, as introduced by Brown and Levinson (1987), examines the verbal strategies that are adopted by the participants in a social interaction in order to minimize the potential effect to their own and the others’ face. In order to better understand and analyse interactions, face is further categorised into positive and negative. It is helpful to clarify that the terms *positive* and *negative* in this context do not refer to good and bad aspects of a person’s behaviour, but rather to the direction (of needs): *positive face* is related to a person’s need for social approval, whereas *negative face* is related to a person’s need for freedom of action (Brown & Levinson, 1987). During any interaction there is interplay between the face wants of the participants, especially since the satisfaction of one’s own face wants is, in part, achieved by the acknowledgement of those of others. Some acts inherently threaten face: orders and requests, for example, threaten negative face, whereas criticism and disagreement threaten positive face. The previous hold for all kinds of social interactions, including those that take place in a classroom, which are full of questions, requests and orders. The speaker therefore must avoid such acts (which may be impossible for various reasons, including concern for her/his own face) or find ways of performing them whilst mitigating their face-threatening effect—i.e., making them less of a threat. Brown and Levinson (1987) have proposed a taxonomy of available strategies to achieve this, varying from avoiding the intended action (e.g., by remaining quiet) to explicitly expressing it without any concerns for the other’s face. In between we find verbal acts containing some redressive actions, which are a way of indicating that no face threat is intended. In this case, various forms of indirect and vague language are often preferred. A more detailed description of these categories is presented in the next section.

4 Connections Between Politeness and Positioning Theories

Early positioning theory work makes connections to the concept of ‘footing’ described by Goffman (1981), and thus has some relationship to politeness theory, which was described in the previous section. Goffman’s analysis of interaction deconstructed the two “folk categories” of speaker and hearer into smaller elements, and he considered different ways for people to listen, to speak, to overhear, etcetera. These different ways of participating in interaction were identified by Davies and Harré (1999) as bearing resemblance to what they called ‘storyline.’ However, they criticized Goffman’s apparent inability to “escape the constraints of role theory” (p. 45), arguing that “the whole ‘apparatus’ [of positioning, and storyline] must be immanent, reproduced moment by moment in conversational action and carried through time, not as abstract schemata, but as current understandings of past and present conversations” (p. 45). Nevertheless, the most explicit connection between the theories lies in the interest in moral orders. Kádár and Haugh (2013) identified how “politeness constitutes a social practice because it involves evaluations that (implicitly) appeal to a moral order: a set of expectancies through which social actions and meanings are recognisable as such, and consequently are inevitably open to moral evaluation” (p. 6). Similarly, positioning theory was described by Harré and van Langenhove (1999) as the “study of local moral orders” based on ongoing shifting patterns of “mutual and contestable rights and obligations of speaking and acting” (p. 1).

The radical focus on the immanent described by Davies and Harré (1999) has been contested by Herbel-Eisenmann, Wagner, Johnson, Suh, and Figueras (2015), who identified positioning theory’s concept of storyline as a connection to larger, relatively transcendent discourses. Their recognition of a dynamic relationship between communication acts and storylines underpinned the authority framework that we are using in this analysis. With this view of positioning theory, it comes quite close to the conception of footing and frame in politeness theory. Both of these theories are interested in moral orders to describe interactions. However, we identify a difference in the way these theories’ traditions do analysis. With politeness theory, analysis generally begins by identifying categories of communication acts, and then relating those to moral order. With positioning theory and the authority framework used in this paper, analysis generally begins with identifying clues in the interlocutors’ sense of obligation.

5 Context and Methodology

To focus on the juxtaposition between the two conceptual frames we draw on the observations and conclusions from an earlier analysis (Wagner & Herbel-Eisenmann, 2014) of a teacher responding to and managing authority relationships when changing schools and thus moving from a familiar context where he was comfortable and established in a small school to an unfamiliar context with different demographics in a much larger school. In order to analyse the transcripts, we use mainly two analytical frames that derive from the approaches mentioned before.

The first analytic frame is the one used by Wagner and Herbel-Eisenmann (2014) in their initial analysis of the context described above. They used a frame that emerged from their analysis of classroom interactions as represented in a large body of transcripts from secondary level mathematics classrooms (Herbel-Eisenmann & Wagner, 2010). They distinguished among four authority structures, which are outlined in Table 1, along with guidelines on how to identify these structures in mathematics classroom interaction.

The second analytic frame is based on the premises of politeness theory and the relative notion of face. As mentioned, during any interaction there are instances when the speaker has to make a choice on whether to perform a face-threatening act or not. One choice is to not perform the act at all by remaining silent or by changing the topic of discussion. The second choice is to perform the act, and then there are two options:

(a) go off-record, i.e. do not express the act directly, but implicate it;
(b) go on-record, i.e. express the act either directly or by accompanying it with a redressive act (aiming to the other's positive or negative face).

In the cases when the speaker wants to avoid a face-threatening act (including the acts that threaten her/his own face), there are some verbal tools to be deployed: hedges are "words whose job is to make things fuzzier or less fuzzy" (Lakoff, 1973, p. 490) and their use in the mathematics classroom may lead to some very interesting utterances, such as the following:

> The maximum will probably be, er, the least'll probably be'bout fifteen. (Rowland, 2000, p. 1)

The 14-year-old student who has uttered the above apparently does not conform to the sociomathematical norm of clarity: a mathematical expression is expected to

Table 1 Analytical guide for authority structures from Wagner and Herbel-Eisenmann (2014)

Authority structure	Linguistic clues	General indicators of the structure (that may not involve the particular linguistic clues previously identified)
Personal authority	• *I* and *you* in the same sentence • Exclusive imperatives • Closed questions • Choral response	Look for other evidence that someone is following the wishes of another for no explicitly given reason
Discourse as authority	• Modal verbs suggesting necessity (e.g., *have to*, *need to, must*)	Look for other evidence that certain actions must be done where no person/people are identified as demanding this
Discursive inevitability	• *going to*	Look for other evidence that people speak as though they know what will happen without giving reasons why they know
Personal latitude	• Open questions • Inclusive imperatives • Verbs that indicate a changed mind (e.g., *was going to*, *could have*) • Constructions that suggest alternative choices (e.g., *If you want*, *you might want to*)	Look for other evidence that people are aware they or others are making choices

be unambiguous. However, it appears that he did it in order to protect his positive face, in case his answer was wrong. At this point, we need to stress the fact that any verbal act may have an effect on both the speaker and the hearer's faces: when a teacher asks a question (knowing its answer) s/he does it in order to maintain face; in other words, to sustain his/her position as an authority in the classroom. At the same time, asking a question is a face-threatening act: it poses a threat to the student's (negative) face. These considerations lead to the following categorisation of verbal acts (Tatsis & Dekker, 2010), which are an expansion of the typical face-threatening versus face-saving dichotomy:

- *face-threatening act*: explicitly threatens the other's face (e.g., requests, orders, rejection of the other's suggestion, expressions of sarcasm and irony);
- *face-empowering act*: explicitly or implicitly empowers the other's face (e.g., acceptance of the other's suggestion, expressions of appraisal);
- *face-weakening act*: implicitly weakens one's own face (e.g., expressions of uncertainty, withdrawal of one's own suggestion, admittance of being mistaken);
- *face-maintaining act*: implicitly aims at maintaining one's face, even when it is not being explicitly threatened (e.g., initiation of talk, expression of one's ideas);
- *face-saving act*: aims at 'repairing' one's face after having received a face-threatening act (e.g., argumentation, justification of one's own acts, repetition or elaboration of a suggestion, expression of face-threatening acts against the other).

In the following section we present our juxtaposed analyses of the same transcripts; we aim to demonstrate that our approaches are compatible and, moreover, they complement each other.

6 Complementary Analyses

The following excerpt was taken from Mark's initial classes in the new context. This was the earliest transcript available in that context. Mark was leading the class in the prime factorization of 72. They had $72 = 3 \cdot 3 \cdot 2 \cdot 4$ so far. (Participant names are pseudonyms.)

a134 Mark:	In order to perform the prime factorization we have to break it down so that all the factors are prime numbers. So as of right now, we have three of our four numbers are prime numbers, correct? So keep working. So now we have, "Two times what are the factors of four?"
a135 Alexis:	Two times two
a136 Mark:	Two times two. Two times two is what the four was. And then we have our times three times three. Of the five factors we have now, how many of them are prime?

a137 Students:	All
a138 Mark:	Okay, if we look back over here, "Two times two times two times three times three times three." That's how we get from seventy-two. This is how we perform our prime factorization. Okay. So that's why I was saying it's not expected that you know that this right away is the prime factorization of that
a139 Simone:	Where would we need, where would we use a question like that?
a140 Mark:	You are going to use it later on. It makes it very easy later when we are cancelling out or dividing by numbers
a141 Jerry:	No, what's a job where we would need
a142 Mark:	What job? Uh, not everything we do in math in high school is going to give you, uh, is going to be used in everyday life. Okay. Everyday life you do some adding, subtracting, multiplying and dividing, right? Okay
a143 Emily:	I sleep
a144 Mark:	You sleep. You don't spend any money? Okay, anyway the purpose of our math courses is to give us all the tools that we need, right. So that later on when you decide on a career that you want to do that you have all opportunities open to you
a145 Kate:	What if you want to have nothing to do with math?
a146 Mark:	Oh everything has to do with math
a147 Jordan:	What if she wants to work at McDonalds?
a148 Mark:	Money, money, money is math, math, math
a149 Students:	[Many students are talking.]
a150 Mark:	All right. Back to the rules of mathematics. Back to the land of the living. Okay, I want you to find all the prime factors of thirty-two. Thirty-two, prime factors of thirty-two. Use your divisibility rules if you're stuck

Wagner and Herbel-Eisenmann (2014, p. 879)

In this transcript, Wagner and Herbel-Eisenmann (2014) found evidence of all four kinds of authority structures in their framework—*personal authority*, *discourse as authority*, *discursive inevitability*, and *personal latitude*. Their analysis was arranged by the categories, not by the progression of the dialogue, but in our comparative analysis here, we follow the progression of the dialogue.

The episode began with Mark telling students what to do. For example, he used the imperative "keep working" (turn a134) referring to their work on the question that he had given them earlier. This represents *personal authority* because he did not give reasons for this task; his expectation for them was based on his authority or status. The same turn also bears evidence of *discourse as authority* because Mark noted, "we *have to* break it down". By the plural pronoun "we" Mark associated himself "with some other (un-named) person or persons, thereby appealing to an anonymous 'expert' community to provide authority for the imposition of a certain kind of classroom practice" (Rowland, 1999, p. 20). This indicated that some rules outside the class (probably the mathematics discourse) required certain action.

With a politeness theory lens we see each of these kinds of explicit demands as an *order* or *request*. Thus, they threaten students' negative face because they constrict action. This threat would normally be accompanied with a redressive action. In this vein, by presenting the request in the *discourse as authority* form, Mark minimises the possibility of disagreement by aligning himself with the students, submissive to the external authority. This may be seen as a redressive action or as a rationale for no redressive action.

We note that in many mathematics classrooms there is no immediate redress of such threats on face and this results in tension. In this case there were a few more turns that did not seek redress, but the situation changed in turn a139, when Simone asked, "Where would we use a question like that?" Wagner and Herbel-Eisenmann (2014) identified this as an example of *personal latitude* because the act of asking a question is evidence of a choice made by the student. Moreover, because Mark had told the researchers that he wanted his new students to ask questions like his students had in his previous school, Wagner and Herbel-Eisenmann's analysis suggested that he did not take Simone's question as a challenge to his authority. However, the students in Mark's class would not have been privy to this information.

By contrast, a politeness theory lens sees Simone's question as threatening to Mark's positive face. Although the students were encouraged to asked questions, the content of the particular question had nothing to do with the topic under discussion (factorisation) but with the usefulness of the task. Thus, Mark's authority, and the authority of mathematics (represented by Mark) was questioned, and that could be the reason why Mark answered Simone directly in turn a140 saying "you are going to use [factorisation] later on." Wagner and Herbel-Eisenmann identified this answer as an example of discursive inevitability because "you are going to" suggests only one possible future event and this event was beyond the control of Mark or his students. Using a politeness lens, however, his answer could be taken as Mark redressing the threat to his face. His reply did not treat Simone's question as out of place, and so he did not threaten her face further. Answering her question suggests openness to her contributions, so it avoided the threat of negative face, and the answer also recognized her authority and thus avoided threat to her positive face. However, the discursive inevitability and the only one possible future event it suggests is immediately challenged by Jerry in turn a141 who moved the topic of discussion again from mathematics to its usefulness in the students' everyday lives and potential jobs. It seems as if the students were attempting to establish a new orientation to the situation, in which the content of teaching is valued according to its usefulness to their future lives (as perceived by them). We suggest that this prioritization of future potential can be viewed as a sociomathematical norm striving to be established.

The section from turn a139 to a149 was described by Wagner and Herbel-Eisenmann (2014) as an example of personal latitude because the students were asking questions and Mark was answering them. Politeness theory, on the other hand, highlights the tensions in the classroom: the students persisted in their line of questioning, not accepting Mark's answers, thus trying to re-establish the definition of the situation. This continued the threat to his positive face. Mark continued to mitigate the tension by performing some face-saving acts. He answered but not in a way that questioned the students' authority to question him. Wagner and Herbel-Eisenmann (2014), later in their article, identified this as a tension between Mark and his students. The students were not accustomed to his preferred way of interacting with them. Their research reporting said Mark's actions broke what the students thought to be social norms for a mathematics classroom. Thus in a way it was a threat to the students' positive face. For example, in Emily's case (turn a143) he replied in a rather sarcastic way by repeating that the student usually sleeps and does not do anything related to mathematics (e.g., spending money). But immediately, Mark realised the threat to the student's positive face and relocated the focus of the talk from the personal (you) to the social (us), and also by using the hedges "Okay, anyway". When this discourse turned into a buzz (turn a149), Mark exercised his personal authority and cut off the students' autonomous questions. This is an example of imposing a definition of the situation by threatening one's negative face baldly, with no redressive action. We have seen that even preservice teachers, when, for example there is time pressure, do exercise their authority baldly (Tatsis & Maj-Tatsis, 2017).

The issues raised during the above interactions led Mark to frustration. This, in turn, opened up for him the possibility to discuss with his students the topic of authority (Wagner & Herbel-Eisenmann, 2014), which can also be seen as an attempt to reach a common definition of the situation. During this discussion he referred to "the authority" as the holder of knowledge and he also stressed the different sources of authority that vary from him (as a mathematics teacher) to each and every one of them (the students). He followed with a couple of examples that he expected would illustrate the nature of mathematical authority. In particular, he asked the students which of two expressions written on the blackboard was correct: $2 + 3 \times 5 = 30$ or $2 + 3 \times 5 = 17$. The vivid discussion that followed resulted in a shift in the authority structures. Wagner and Herbel-Eisenmann reported that the students started exercising personal latitude by making demands of Mark. This resulted in some moments of tension, which politeness theory identifies with face-threatening acts:

b150	Mark:	If we follow as what you guys…or what we refer to order of operations we're following what?
b151	Cam:	Is this a trick?
b152	Mark:	No, it's not a trick.
b153	Cam:	Yah, it is. What about this one? Which one?

The above excerpt contains an interesting reaction of a student to Mark's question. Instead of directly replying to Mark's request, Cam made a meta-comment on it, expressing (and repeating in b153) that the request was actually "a trick." Cam may have been afraid that whatever answer he gave would be wrong, in which case he may have wanted to protect his positive face from the potential threat of a wrong answer. Perhaps the student had a similar experience from a previous mathematics teacher, or, in other words, maybe this was an already established sociomathematical norm. In any case, he chose to respond to the face-threatening act with a meta-comment, which could be interpreted as a face-threatening act to Mark. At the same time, Cam showed his non-adherence to the norm that a teacher may ask "tricky" questions.

After the students discussed the order of operations for a while, Mark referred them to a polynomial expression and asked them why the x-terms cannot be added to the y-terms. Ashley said that x and y are different numbers, and Mark replied with the same question—why can't they be added?

b202 Ashley:	Because you told us yesterday.
b203 Various:	[*Inaudible too many voices*.]
b204 Brienna:	You contradict yourself.
b205 Mark:	I'm not trying to contradict myself.
b206 Brienna:	Yes, you are. You're like, "Oh well, why is it like that?"
b207 Mark:	Okay. Shhh.
b208 Brienna:	I am not Albert Einstein.
b209 Mark:	How do you know?
b210 Brienna:	Uh, do I have his hair?

The above excerpt presents another situation in which Mark had to deal with a bald threat to his positive face. He responded in that threat in a way that did not pose any threat to the student's face. In this case, Ashley claimed that Mark had told them that during the simplification of an expression one should not collect the x- and y-terms. Mark responded to that by an indirect evaluation; he went off record by choosing to not express the face-threatening act of the negative evaluation of Ashley's idea. Instead, he said that he was curious on the roots of her idea. Mark's choice led to Ashley replying that it was him who had expressed that rule in the previous day (b202). This then led to Brienna's direct face-threatening act: "You contradict yourself." She noticed that he gave them a rule one day and then questioned the rule the next day. Mark felt obliged to immediately refute this threat (b205) in order to protect his positive face: a good teacher is not the one who contradicts himself. Brienna insisted in turn b206, by adding a very interesting meta-comment on Mark's discursive strategy to ask students to justify their opinions: "You're like, 'oh well why is it like that?'" At that moment, Mark probably felt that his intended definition of the situation, as well as the supporting norms are at a serious risk; that might explain his request for silence. At the same time, it is noteworthy that he (deliberately) chose to not perform an explicit face-saving act. Brienna's response was notable; after realising that she had "crossed the border" with her face-threatening acts towards her teacher, she chose a face-saving act: she aimed to justify her behaviour by stating that she is not Albert Einstein. In other words, she stated in a humorous way that Mark should not expect her to have all the answers to his questions. Humour functions here as a tension-release mechanism, or as a redressive act. This point is crucial: the students (in this case Brienna) seemed to be unafraid to take the responsibility of questioning the definition of the situation that was promoted by their teacher (threating his face), and they also took the responsibility to protect his face. As Wagner and Herbel-Eisenmann (2014) state "When Mark challenged his students with questions about authority, they exercised authority by telling him how they wanted him to teach them" (p. 881).

Another way to interpret this shift is by focusing on the establishment of a social norm: the student has responsibility for learning and also may question the teacher's approach. This in turn resulted in a shift of the way that face concerns are considered. In a shared-authority interaction, a request—even expressed baldly—still carries a potential threat to the addressee's face, and might not require a redressive action by her/him. This frees more space for productive exchanges and an effective focused interaction (Goffman, 1972).

7 Discussion—An Attempt to Reconcile

Both of the frameworks used to analyse the episode here are frameworks that relate to authority. The Wagner and Herbel-Eisenmann (2014) framework explicitly identified four "authority structures" that were grounded in pervasive word patterns in mathematics classrooms: personal authority, discourse as authority, discursive inevitability, and personal latitude. Politeness theory refers to people's need for freedom (for negative face) and social approval (for positive face) and identifies the strategies deployed in order to minimize the threat to one's and/or the other's face. Additionally, all verbal acts can be categorised according to their effect on the speaker and/or the hearer's face. The two frameworks highlight different aspects of authority. For example, changing one's mind is seen as an expression of personal latitude if interpreted with authority structures, whereas from the face wants point of view we need to examine the actions that led to it; the person who changed his/her mind might be merely following the request (implicit or explicit) of a person being *in* authority or being *an* authority. Moreover, changing one's mind could be seen as a face-weakening act, depending on the context. This pluralism of interpretations should not be seen as leading to conflicting results. The teacher's authority is an aspect of her/his positive face, whereas students' positive and negative faces also are in play during any interaction in the classroom.

Our complementary analyses of the episode have provided some interesting interpretations on the interactions of Mark with his students. Just as with most interactions in the classroom, there were moments of tension between the participants. These moments usually signify the points when the existing or the proposed definition of the situation is questioned. This definition is in turn based on some social and sociomathematical norms. We have seen at least two norms being established in Mark's classroom: the first is related to the usefulness of school mathematics (sociomathematical norm) and the second with the shared responsibility on the nature of teaching—or, particularly, questioning (social norm). With the help of Mark, his students have shown their willingness to participate in the joint establishment of an acceptable definition of the situation. Furthermore, we have seen instances when a 'typical' school norm is not adhered to by the students:

the teacher is expected to ask questions and the students are expected to reply. The degree of conformity to a norm is a significant factor of the ongoing definition of the situation; in other words, it shapes the classroom learning community.

Attending to face wants may help a teacher be responsive to students. As soon as the teacher identifies the face wants of the students s/he can accordingly modify his/her actions to either open space for action (redressing negative face) or affirm the students' authority or status (redressing positive face). Indirect requests and the use of hedges are two possibilities; expressions of appraisal are face-empowering acts (Tatsis & Dekker, 2010), thus may empower the student's authority. In any case, moments of tension do arise in the classroom and eventually the teacher has no other option but to exercise her/his authority (see turns a150, b207) in order to lead the class to his/her expected definition of the situation. We have seen in other research that this behaviour can be observed even in preservice teachers (Tatsis & Maj-Tatsis, 2017); moreover, it may eventually assist the teacher and the students to reach a common learning goal, especially when students do need a scaffold in order to proceed.

These observations using politeness theory frameworks to describe the ongoing negotiation of the social situation are reminiscent of the way positioning theory describes the negotiation of positioning—distinguishing among first-order, second-order, and third-order positioning (van Langenhove & Harré, 1999), which refer to the degree of explicitness of such negotiation. The authority framework developed by Herbel-Eisenmann and Wagner (2010) rested on positioning theory but does not highlight the shifts of positioning. Rather, it focuses on the presence of particular positionings.

The politeness theory in this analysis helped elaborate aspects of the analysis using Wagner and Herbel-Eisenmann's (2014) framework. In particular, the three authority structures that feature demands (personal authority, discourse as authority, and discursive inevitability) threaten the face of students. A teacher's choice to embrace the two structures that position him- or herself along with the students' subject to the discourse (discourse as authority and discursive inevitability) mitigates the threat. This may help explain why mathematics teachers gravitate to these authority structures. Furthermore, the authority structure called *personal latitude* may seem the best for teachers wanting students to develop authority, thus empowering their face. However, politeness theory explains why this structure is full of tension, especially at moments when the existing definition of the situation is being questioned. This tension may again explain why teachers gravitate to the other authority structures.

We follow with a table that gives an overview of the authority structures in terms of positioning and politeness theory orientations (Table 2).

Summing up, the two theories are both interested in the phenomena that occur during classroom interactions; moreover, we see them as complementary since politeness theory helps us consider reasons for teachers and students to choose particular authority structures in their classroom interactions. Thus, we generally

Table 2 Insights from politeness theory in relation to positioning and authority

	Positioning	Politeness
Personal authority	Teacher guides, students follow (in mathematics, experts tell us what to do)	Face wants of teachers may threaten the face of students
Personal latitude	Teachers and students choose and explore mathematical tools and methods (people make choices in mathematics)	Face wants of students empower their own face but may threaten the face of teachers
Discourse as authority	Teachers and students follow (mathematics tells us what to do)	Redresses or avoids face challenges by aligning teachers with students, both subject to mathematics, but may challenge their face by denying their negative face wants
Discursive inevitability	Teachers know what to do, students develop this knowledge (with mathematics we know what must be done)	

believe that in order to fully comprehend the dynamics of the exchanges in the mathematics classroom we need to be able to continuously shift our focus from the participants' acts to the established (or striving-to-be-established) norms and from the participants' positionings to their own and the others' face-wants.

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The Use of Language in the Construction of Meaning for Natural Number

Lorena Trejo-Guerrero and Marta Elena Valdemoros-Álvarez

Abstract This study reports the results of research on the construction of meaning for natural numbers, through the development of division in a primary classroom. The teaching sequence contrasted two different approaches to division with natural numbers, canonical division and Egyptian division. The analysis of the case study of one teacher, Karina, focuses on the significance of mathematical language in one lesson.

1 Introduction

We consider language as one of the most important tools that allows verbal communication among people, by which we interpret and translate the everyday experience; but it is necessary for a proper exchange of meanings to own a language code required for each area of social interaction (in this case, the code of arithmetic language).

In this qualitative study, we present a case on the teaching of division with natural numbers grades in primary school. We examined how students, relying on the use of language, acquired several notions and concepts related to division of natural number. We present the case of one teacher, Karina, analyzing one lesson on division with natural numbers, contrasting Egyptian and canonical division. The analysis describes how the construction of concepts depended on the meanings of words and these depended on experiences connected to teachers' and students' previous knowledge.

Karina discussed Egyptian division in her lesson to lead students to compare it to canonical division and to support transfer of meaning from the first to the second. Hereinafter and in particular in the next section of this chapter, we

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© Springer International Publishing AG, part of Springer Nature 2018

J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_14

designated as *Egyptian* division the old North African procedure as identified by Guillings (1982), Ifrah (2000), Barradas (2004), Van Der Waerden (1975), among others. We used two different methods for dividing natural numbers, canonical division and Egyptian division. The rationale for using these two methods was to allow teachers to reflect on the different ways to develop meaning for division, using operations like multiplying by 2 and dividing by 2.

Our research questions are the following: (1) How did the teacher design a lesson to support student comprehension of natural numbers division? (2) How did students use representations to explain the solution process they had used?

2 Theoretical Framework

Egyptian division, as posed by Van der Waerden (1975), summarized by Guillings (1982) and Barradas (2004), consisted in 389 divided by 19. Take the number 19 and form two columns as follows: in the first row write 19 (divisor) and 1. Next, rows appear by doubling both elements until obtaining in the column that corresponds to the number 19, a number whose double exceeds 389 (Fig. 1).

Then, decide which numbers from the first row are possible to add bottom up without exceeding 389, in this case 304 + 76. The result of the division is then the sum of the corresponding elements in the right row (in this case, 16 + 4 = 20). As 304 plus 76 is equal to 380, we know that the remainder is 9 (Fig. 2).

This splitting or partial duplication principle, which is used in multiplication and division operations, eliminates the need of learning or building multiplication tables (Collette, 1986).

Fig. 1 Egyptian procedure for division (Barradas, 2004)

	Dividend	Factor	
	389	19	
Double	19	1	Double
	38	2	
	76	4	
	152	8	
	304	16	

We stop doubling when the double of the last value of dividend exceeds 389.

Fig. 2 Results of the Egyptian division procedure (Barradas, 2004)

	Dividend	Factor	
	389	19	
Double	19	1	Double
	38	2	
	* 76	4*	
	152	8	
	*304	16*	

We add from below to above in the column of duplications of the dividend to find exact or slightly less than the dividend 389. *(304 + 76 = 380), *(16 + 4 = 20).

2.1 *Thinking and Language*

Vygotsky (1993) assumed that knowledge is generated in a social and culturally organized environment. Both psychological research and teaching practice often have considered that knowledge is independent from the context in which it is acquired, and once specific knowledge is constituted, it can be applied to any situation. For instance, if a student knows how to add and subtract, he/she will be able to solve any problem that occurs in any situation concerning addition and subtraction, both in school or in everyday life. However, research has shown that people who fail in homework and school mathematics tests can be competent in everyday situations that involve the same mathematical calculations.

Vygotsky dealt with speech and thinking to the extent that speech is involved *in communication* of knowledge between people (Slobin, 1974). Vygotsky conceived of language as a mediator between the individual and culture; students may acquire a scientific concept addressed by the use of words and the main role of school is to help them to systematize those concepts that children already own (D'Amore, 2014), and which have been incorporated in to their knowledge little by little from their cultural environment.

To analyze the language that students use in classrooms we refer to Cazden (1991), who considers that in a conversation between equals, it is when children interact with their peers that they provide intellectual elements by means of posing questions and answers, which does not occur in a conversation with a teacher. For discourse between teacher and children in classroom, we use Shuard and Rothery's ideas (1984), with regards to the flow of meaning, according to uses of the teacher's language and its re-signification by children.

We consider speech as a tool that helps to facilitate the acquisition of linguistic competences. It is also a symbolic instrument by means of which each individual organizes both the environment and their thinking; at the same time, speech develops in a different way when it is oral than when it is written; specifically as a cultural resource and pedagogical communication (Bourdieu & Passeron, 1970). Thus, one of the roles of school is that children understand texts and the sense of verbal interactions, which leads us to analyze the linguistic interchange produced in the classroom.

For the above, we use two proposals (Thompson, 1993) for discourse analysis: (a) *conversational analysis*, whose methodological principle is to study examples of linguistic interactions in the real scope as they occur and to pay careful attention to the way these are organized; (b) *argumentative analysis*, according to which discourse involves reasoning chains that can be reconstructed in many ways, and in the case of elementary mathematics are interpreted as inference patterns that address one issue or topic to another.

2.2 The Teaching of Division with Natural Numbers

In the teaching of division with natural numbers it is necessary to identify the effect of teaching decisions as the manifestation of a personal style. According to their beliefs (both mathematics and didactics), each teacher proposes to the group a particular strategy because they consider it appropriate. However, what is good for one student may not be good for another; and thus such teaching strategy becomes a *didactic obstacle* as D'Amore (2005) states. The above confirms that any re-signification given by teacher to Egyptian division will have to be adapted to the characteristics of each group of students.

Reflection on the teaching of mathematics occurred in college working groups, whose goal was interchange and propose new ways to guide problem situations in order to transform them into opportunities that allow teachers to know their restrictions and construct alternative actions that lead and improve teaching practice. Teachers' systematic observation of events in the classroom were a necessary tool to discover new ways of addressing division with natural numbers that allow teachers to reflect along with colleagues about their experiences with students, when they dared to express the difficulties involved in the understanding of mathematics notions and concepts, such as division with natural numbers. The discourse analysis allowed us to approach those students' conceptions about a specific topic and also teachers' conceptions about those same concepts; different studies were carried out to link these two points.

2.3 Lesson Study and Language Use

Lesson study and observation in class is an activity that promotes the improvement of teaching. Respect among all teachers is important, both to express ideas, concerns, questions and explanations and to listen to the other, share experiences, contribute with ideas, beliefs, knowledge and commitment to improve the teaching, hoping that efforts will be reflected in the following generations (Isoda, Arcavi, & Mena, 2007).

We encourage teachers to discuss and express their views collectively, along with their colleagues at school or school zone that allow them to make conjectures and experience the process of doing mathematics and to interact with the disciplinary and educational part of this discipline, and subsequently be able to teach students wherever they are; as well as to propose and be critical and creative.

One of the most relevant skills that children in basic education need to improve is the ability to express solutions of the arithmetical problem for themselves. The *mathematical thinking and language* are used in several mathematics activities with many types of content (Isoda & Katagiri, 2012). The development of mathematical reasoning skills helps to understand and solve problems, since the arithmetical-technical language related to such thought allows each individual to face and act before problems in everyday life.

2.4 Shared Thinking

Analysis of the construction of shared meanings, both with students in classroom and in college sessions, helped us to develop a recreation of the educational reality in rural schools, understanding as knowledge-sharing in school environments, the academic knowledge of the teacher, regarding the topic to be taught, and the previous knowledge of the student, concerning the curricular content to learn. The role of language becomes the main element of objective because it is through this process that new conditions related to the environment arise.

The structure of speech in the teaching and learning in primary school shows how the conventional social order in classroom incorporates a linguistic order, that is, the structure of speech represents the ways education take place in these contexts. In this manner, when studying discourse sequence, we can carry out an empirically detailed study: how can teachers partially select their knowledge to present to students? How do teachers separate topics and re-arrange them to pass on to students? which brings us to analyze the way of what is said and its content (Edwards & Mercer, 1988), to understand what happens in the school classroom.

3 Method

3.1 A Case Study Related to Lesson Study

This chapter details a case study that discusses how one teacher, Karina, incorporated mathematic technical language into a lesson on natural number in situations of division, with the canonical algorithm and the Egyptian procedure.

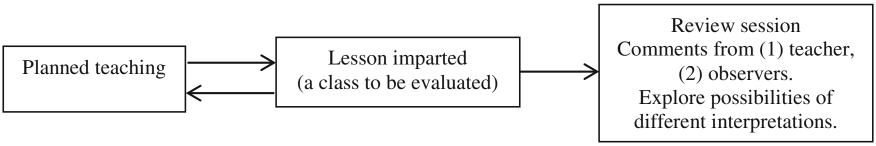

Fig. 3 Lesson Study according to Isoda, Arcavi, and Mena (2007)

For the Lesson Study, a group of teachers participated and collectively designed a lesson about the procedure of canonical division and that of consecutive doublings in the Egyptian division procedure. One teacher in this Lesson Study group, Karina, was chosen to carry out the lesson planned during a collective session of teachers. Karina was in charge of a group of fifth and sixth graders in a multi-grade elementary school. The rest of the teachers participated in that lesson as observers.

The purpose of the review session was to explore ways to improve the lesson, analyzing in particular the development of language and improvement in the sense of division, in the transition from canonical division to Eygptian and its inverse, as in fact happened in the classroom. To summarize the work that preceded the lesson, we present in Fig. 3 some episodes of the Lesson Study identified by experts (Isoda et al., 2007, 31).

3.2 Research Participants

We selected teachers from multi-grade and complete organizations schools (schools with one teacher per grade). We visited Zone 105, in Cuyamaloya (State of Hidalgo, México) which covers 12 schools; only two of them correspond to complete organization and the other 10 are multi-grade, where 3 or 4 teachers work in each school. That is why teachers teach two grades each (the whole elementary school consists of 6 grades); school routine is very different in each school, although they are geographically very close one to another.

From the above public elementary schools, we chose one complete institution and two multi-grade schools that belong to the Education Ministry of Hidalgo State. In the three selected schools we worked with all teachers that belong to this School Zone, who participated as observers (including the school supervisor, the technical-pedagogy supports and principals of the schools).

Participating teachers' ages ranged from 28 to 50, and their students in fifth and sixth grades were 10 and 12-year-olds. Teachers' profiles varied in training and years of work experience. We chose three teachers: Marco, Luna, and Karina to carry out the case studies.

Teachers in the Lesson Study have different teaching and work experiences: Marco has worked in complete-organization schools; Karina has worked in multi-grade schools; Luna has worked in both types of schools. These features in teaching show three different settings that are usually present in the rural school in the State of Hidalgo, México.

3.3 Tools

We designed tools for each task. The approach of the study of cases states that there are three kinds of educational curricula: the projected one, the involved one and the achieved one (Isoda & Olfos, 2009. This research study is focused on lessons about the teaching of natural numbers in fifth and sixth grades of elementary school during three moments: (1) an initial session with in-service teachers; (2) teachers and students in the classroom, and (3) analysis sessions with teachers after classes. Once we finished our first working sessions with teachers and students, we followed a protocol where teachers could write what they considered relevant to be reviewed in the lesson plan and those events that occurred during class, which were used in a feedback session to guide teachers' reflections on the lesson. In joint sessions, teachers pondered the importance of a visual organization of the information and use of whiteboard during problem solving situations.

3.3.1 First Joint Session with Teachers

In the first joint working session (first phase of Lesson Study), teachers analyzed the contributions provided by the history of Arithmetic to compare our accustomed way of multiplying and dividing to the way Egyptians and Russians (centuries ago) used to do. We also presented features of natural numbers and elemental arithmetic operations with such numbers, and some relevant features of arithmetic properties in elemental operations and their reversibility relationships. We also talked about the importance of the teachers' generalization process, which allowed them to design arithmetic problems with natural numbers as well as to prepare a plan for a lesson that could be observed by participant teachers.

Concerning the observation of the joint group of teachers, we considered that the lesson Karina designed, served to encourage and reiterate our bases about the mathematical content previously covered. To the extent that the group analytically returned to the topic of the lesson that originally had an abstract view (to compare canonical division to Egyptian division), little by little it became an instrumental and concrete task. The dialectical spiral expresses the sense that teachers grant to what they see to be able to conceptualize the sequence of the lesson in terms of the evolution of knowledge collectively shared. In other words, to transmit this knowledge with new language that serves as a link to a new turn of the dialectical spiral that promotes new re-interpretations in the development of a lesson, as stated by Pichón-Rivière (2009) and coinciding with Bauleo (2009).

3.3.2 The Design of Mathematics Lessons

All mathematical argument has its own epistemological status which depends on the history of its evolution within mathematics, its critical acceptance in this field of knowledge, those reserves of their own, and the language that expresses or that is required to express. For instance, when in the history of the evolution of a concept a non-continuity or fracture is identified, radical changes in its conceptions, then it is supposed that concept has epistemological obstacles within to be learned (D'Amore, 2014).

In the second phase of the adaptation of the Lesson Study, we observed interventions related to solving problems with natural numbers and Egyptian division (in the case of Karina). Our attention was focused on Karina's discourse in the classroom; how she used both materials and the whiteboard, and how she evaluated her students. We videotaped classes. Teachers observed and wrote about any difficulties the teacher and students faced, as well as their opinion about these, and also comments that were helpful in the review session.

We programmed a joint meeting with observers (teachers) that took place after observation in class (third phase of Lesson Study), which facilitated reflections and comments of the group about the features of the annotations related to the observation protocol. We listened to and analyzed teachers' arguments.

In the case of Karina, we observed her class in a public elementary school—monolingual and multi-grade—where she works in Cuatro Palos, Singuilucan, Hidalgo State, Mexico. In her classroom there were 22 students, 10 of them in fifth grade and 12 in sixth grade. Nine teachers participated in the observation in class and carried out reflections about the teaching of division with natural numbers. For the study of teacher Karina's class (according to Isoda et al., 2007), we analyzed the school language in two episodes: the first was when Karina taught her lesson to students, as stated by Cazden (1991); and the second was during the joint session with in-service teachers that participated as observers, as described by Isoda et al. (2007). This allowed understanding several aspects of the process of meaning construction shared in the classroom (Edwards & Mercer, 1988).

To validate those meanings that appeared during Karina's lesson, we used as elemental criterion the local meanings of the actions from the teacher and her students' point of view, as well as the observations and reflections of the joint session, because we considered that participants construct meanings based on several versions of the content, depending on the interaction situations.

4 Results

Karina chose the canonical way to divide as a starting point for her class, in order to compare it to the Egyptian procedure because she deemed Egyptian division as a source that causes difficulties for children. She posed *Egyptian* division with the group of teachers who worked in the design of lesson plan, maintained in the

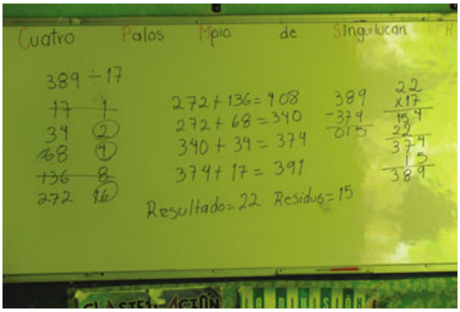

Fig. 4 Example used by Karina

premise that when children use the more elemental operator (in Egyptian division), the sense they give to division could be transferred to the canonical operation by simplifying the code, which itself is a way to re-create the sense. All this was done in the system of decimal numbers (as Egyptians did it in the past).

Concerning the manifested discourse of children, what they and their teacher formulated, Egyptian division was developed in a natural way and all of them used terms such as "remainder" and "result", which are typical for canonical division. At the time Karina guided children to use addition, subtraction and multiplication to verify their results (Fig. 4).

4.1 *Arithmetic Problems with Natural Numbers Posed by Karina*

The proposal included in the Plan and Programs for elementary education (Secretaría de Educación Pública, 2011) aims to take to classrooms mathematics that allows children to construct knowledge through activities that foster their interest, get them involved and keep their attention until finding the solution of a problem. Thus, it is expected that children develop the ability to express ideas, the capability to reason, create and image. In order for children to construct their mathematical knowledge it is necessary that teachers select and design problems that help children to develop notions and procedures. Egyptian division was the alternative that Karina chose to stimulate her students' knowledge.

Karina designed division problems that can be seen in Table 1.

Students solved these six problems through the Egyptian procedure as the teacher requested. One of the features in common is that all problems have a remainder. We assume that Karina wanted children to solve the problem by means of the Egyptian procedure and prove that their answers were correct. In using this procedure students make use of multiplication as the inverse operation to division, and then add the remainder.

Table 1 Division problems designed by Karina

No.	Problem
1	In a video-game, Luis obtained 155 points because he caught 17 equal apples. How many points are for each apple?
2	524 people will go on an excursion, 12 persons will travel in each bus. How many buses will be necessary to drive all travelers?
3	The father of Araceli has to pay $996.00 in 14 payments for a recorder, one per month. How much should the father of Araceli pay, per month?
4	Genaro and his friends produced 907 paper flowers to distribute them in 16 vases to decorate their school. How many flowers will they put in each vase?
5	Raúl helps his uncle to pack 354 apples in boxes of 18 apples per box. How many boxes do they need to pack the apples?
6	Beatriz has 233 candies and wants to share them with her 22 classmates. How many candies correspond to each classmate?

4.2 Work in Teams

When the teacher is clear about the learning goals for a lesson, he/she gives instructions and explanations that foster the implementation of thinking processes, both individual or team, since the resolution of a problem—from a constructivist view—should encourage relationships among students as a mean to improve their learning and personal development, in such a way that children can support each other and learn from others.

The former strengthen the teacher's work considering that the others constitute a reference to learn and children have at the same time the possibility to achieve a balance in their actions, rectify, correct their mistakes or dismiss some activities. In this way, the prevailing role in many models attributed to the traditional role of the teacher in the teaching can be avoided.

Karina continued with her lesson, organized work in teams, and provided them with color printed sheets with the problems solved with Egyptian division (the five problems in the previous section). Students worked with their classmates, discussed and solved the problem the teacher assigned to each team, having on hand the example in the blackboard that Karina wrote.

Prior to students' explanation about the procedure they followed to solve the problem, Karina erased the blackboard and pasted 4 sheets of different color with the text of each problem, in order for each team to come to the front of the classroom, write the procedure and share it with the rest of the class (Fig. 5).

When the last two teams explained their solution process, they pasted a sheet of paper at the bottom of the whiteboard—close to the frame—because the space assigned to each team was not enough, although writing was very small and sometimes illegible for children placed at the back of the classroom. However, due to the verbal explanation of some members of the team, the rest of the class knew what was written on the sheet of paper, and it was confirmed and reaffirmed that

Fig. 5 The students, Karina and observer teachers

teams solved the problem posed in the same way (as the teacher had shown them that Egyptian division should be solved).

While working in teams, interaction among children was more evident; concerning this Karina said:

> Karina: I organized teams with high proficiency students and low proficiency students, so the former may support the later ones. In fact, I write in cards the members of each team, so children know who they are with. I modify teams every two months.

Concerning work in teams, we can mention that each teacher organized the group in small teams, according to what the teacher considered appropriate and the features of each group. This situation is very common in groups of multi-grade modality, where outstanding students work with low students.

4.3 Closing the Lesson

Karina did not question the differences and similarities between the two ways of dividing. So, we concluded that her main objective was that students practice how to solve problems through the Egyptian division procedure.

5 Analysis

5.1 The Multi-grade Classroom

This study took place in a multi-grade school. These are rural schools located in small communities, some of them in regions of difficult access (the only way to arrive is by walking). School registration typically is no more than 20 students per grade that is why 2, 3, or 4 teachers work with students from two or more grades per classroom in these schools.

Teachers' professional experience in this type of schools is different from the experience in a complete organization school, where there are only students from one grade in each classroom. There is communication among teachers and administrative and governing roles are rotated. Technical-pedagogical activities are also organized differently from complete organization schools, since working with students from two grades requires thinking about related content for both grades so that teachers can simultaneously address the needs of two grades.

Regarding the content of division with natural numbers, Egyptian division does not form part of the official curriculum school program. We observed that Karina conceived of Egyptian division as a different way to divide; this allowed her to compare it to canonical division. We also confirmed that she used technical terms that often defined the elements of canonical division in elementary school such as "quotient," "result", and "remainder." She also used basic common arithmetic operations and their formal signs to prove results and to corroborate whether they were correct in both procedures. She pointed out that Egyptian division consists of successive "duplications of divisor and the number one", the procedure stops before one of the duplicated quantities of the divisor exceeds the dividend.

5.2 *The Lesson*

The teacher wrote on sheets of paper the problems students solved in teams, perhaps in order to later contrast their answers.

She carried out a systematic lesson plan, incorporating the use of didactic sources such as the whiteboard and electronic board.

Karina considered Egyptian division as a way of comparing this procedure with canonical division; so she had recourse to conventional basic arithmetic operations and its conventional signs to solve the problem. She used knowledge related to division and about all basic arithmetic operations with natural numbers.

Karina had the following interactions with her group of students: she chose examples for the work in teams; organized pupils in teams, and asked them to share their solution to the problem with the rest of the class.

During work in teams, children engaged in active verbal interaction; they discussed among themselves when solving the problems and came to the front of the classroom to explain how they solved the problem. The teacher asked one of the students to read aloud the text of the problem. In order to continue their presentations of the procedure they had used for the resolution of the Egyptian division, all members of each team participated by mutual agreement. Children re-conceptualized the task and expressed their solutions *in their own language*. For example, students "obtained the double" of the number indicated in the Egyptian division, and we observed that children understood the nature of this type of division by setting successive duplications.

5.3 Joint Working Sessions with Teachers

For Karina this was a unique experience to discuss with her colleagues during the review session of the lesson study. This offered the teacher in charge of the class the opportunity to display a specific way to study content, which propitiated in observers to discuss how each teacher had solved or proposed to solve the same problem. This communication helped to improve a diversity of teaching strategies.

The analysis of the construction of shared meanings, both in class with students and in the group session with teachers, allowed us to develop the educational reality in a rural school. Language became the main element of an objective interaction, since it is through language that children can work with doubling in Egyptian division and at the time they set new relationships and new meanings. At this point, we noticed the teachers' influence in the arguments that children used, with regard to the meaning that canonical division acquires when contrasting it to Egyptian division. Our study about natural numbers by means of division problems, in the case of Karina and the lesson study, facilitated an approach to the real requirements of teachers-in-service.

Observation of the joint group was useful to encourage, in Karina's class, to recover the work with the topic that initially had an abstract character (to compare canonical division to Egyptian division), and then it became, little by little, a concrete, instrumental situation. It is worth mentioning that an expert also observed Karina's lesson, without intervening. Her impartial view served as a link to a new turn of the dialectical spiral towards major specific levels of the observed class, in concordance with Pichón Rivière (2009), ratified by Bauleo (2009).

6 Conclusions

The two moments of the school discourse analysis allowed us to rethink the importance of the didactics of mathematics and its implications both for students and teachers. In the classroom of an elementary school, knowledge is constructed thanks to an interaction process among students, teacher, and content for which it is necessary to analyze not only the constructive activity of the students (early ideas about content, willingness or motivation to learn the content, among others) but also the mechanisms of the influence or pedagogical assistance (this is the role that the teacher plays), as well as the nature of the mathematical content, and the place where mechanisms are produced according to Coll (1992). We add to all this the instrumental use of language.

A common aspect that we noticed was the willingness and challenge that every teacher took when we proposed that they work with other forms of multiplying and dividing, which do not appear in the official curriculum of elementary school. This was undoubtedly one of the core aspects of our research because this led us to recognize both the scope of task designs, such as resignification and communication of knowledge from teachers to students and the re-interpretation that students in fifth and sixth grade achieved about division with natural numbers.

The use of natural and mathematical languages in class—between teachers and students and among teachers—and during the joint sessions in the teachers' working sessions was a core aspect of our study that enabled us to understand the meanings around arithmetic operations with natural numbers: when teachers explained their teaching strategies, the ways students appropriate knowledge, and focus of reflection in the joint sessions of teachers.

We also considered relevant those features that differ in one case from the other, such as the way teachers project their classes, which permitted us to notice that each teacher has their own history of life, both personal and professional, and consequently this affects the way teachers perceive mathematics and its teaching. Thus, the design of a lesson plan is also related to the way teachers understand different ways of multiplying and dividing; how to organize the new knowledge and what to do to prepare and teach (give a class).

With regard to the comparison between canonical division and Egyptian division, we observed this teacher's knowledge about division with natural numbers, because she used signs and expressions of canonical division to explain Egyptian division. The re-signification of Egyptian division given by Karina was adapted to the group of students. The systematic observation of events in a classroom by her colleagues meant that Karina discovered new ways to teach division with natural numbers. We also concluded that reflection with the goal to exchange and propose new ways to study the encountered problems became opportunities to observe her teacher limitations and construct alternative actions that guided and improved the lesson.

We may conclude that systematization of teachers' knowledge according to Shulman (1986) and the creation of reflective habits are necessary to recognize the effects of an educational decision in teaching as the manifestation of a personal style. Teachers carry them out in order to critically understand the personal teaching practice and to modify them starting from actions or plans that respond to the needs of the environment (Bennet, 1979). If this is done in teams, with the idea to exchange and propose new ways to study the encountered problems, these can become opportunities to observe their own limitations and construct alternative actions that address and improve their teaching practice. The systematic observation of the events in the classroom can help to discover the aspects involved in the teaching practice of the teacher and their own personal style.

Two moments of the school discourse analysis (that is, the discourse analysis among Karina and her students and between Karina and the teachers who observed her class) facilitated to pose again the importance of the Didactics of Mathematics and its consequences, both in students and teachers. At the same time, we could recognize the link to the interaction process among students, teacher, and content.

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“I Am Sorry. I Did Not Understand You”: The Learning of Dialogue by Prospective Teachers

Raquel Milani

Abstract This text concerns the process of prospective teachers learning to engage in dialogue with their students in mathematics classes to promote learning. A teaching practice course on mathematics was designed to promote a meeting between the prospective teachers and the concept of dialogue. Investigation, reflection and planning activities were developed to provide such a meeting. The chapter focuses on the dialogue practice of one prospective teacher in the teaching course. Based on these practices of dialogue and theoretical inspirations about dialogue and about interaction, I propose an interpretation for dialogue, whose underlying political stance assumes that the talk is shared by those involved in it. I emphasize the move of going to where the other is in order to understand what she/he says, and also propose some actions that could contribute to the process of learning to be engaged in dialogue.

1 About Communication in Mathematics Education

The types of communication established in the classroom influence the quality of learning mathematics (Alrø & Skovsmose, 2004). Dialogue, lecturing, recitation, the “sandwich” pattern, and quizzing are examples of types of communication that can be found in mathematics classes. However, not all types of communication provide students with an opportunity to express their own ideas.

In many classrooms, it is possible to find patterns of communication where the teacher asks questions for which she/he knows the answer, and the students, in turn, try to guess what the teacher expects as an answer. The teacher evaluates the answers as right or wrong. If they are correct, the conversation is completed by the teacher by saying “very well.” If wrong, a new comment is made, and the student tries, again, to guess the teacher’s required answer. This conversation is characterized as a guessing game, in which the questions are usually posed by the teacher

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© Springer International Publishing AG, part of Springer Nature 2018

J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_15

and the answers are provided by the students. As the questions become more direct and the answers more explicit, the teacher ends up funnelling the students' responses (Alrø & Skovsmose, 2004). Once the teacher is positioned as an authority in the classroom, she/he is the one who validates the answers. Thus, a student's talk always occurs between the teacher's utterances. An example of this is when the teacher asks a question, the student answers, and the teacher, finally, evaluates it. This pattern of communication is known as "sandwich" (Alrø & Skovsmose, 2004; Streitlien, 2010). It is also known as IRF, Initiation-Response-Feedback (Sinclair & Coulthard, 1975), or IRE, Initiation-Response-Evaluation (Mehan, 1979).

The literature shows significant reflections on actions related to dialogue, such as listening to the students' reasoning, posing questions in order to contribute to students' learning, and inviting them to talk about mathematical concepts. Ball and Fornazi (2009) stressed, among other aspects, the importance of the students giving their reasoning explicitly in order for the teacher to get to know these ideas and use them in the development of the class. In this way, the students can construct mathematics together.

According to Wallach and Even (2005), there is not a specific moment in the teacher's practice to assess student understanding, but rather it is an integral part of teaching. "This can be done, for example, by observing students solve mathematics problems in class, by listening to their mathematical discussions during the lesson, by attending to the nature of their participation in class activities, and by being sensitive to their feelings" (p. 396). Hearing the students is a powerful tool to understand what they are saying, showing, feeling, and doing while engaged in mathematics problem solving.

Asking questions is an important action related to dialogue. Considering this action, Moyer and Milewicz (2002) describe some possibilities to prospective teachers to improve this ability. Moyer and Milewicz state that when prospective teachers design a list of question to be used in practices such as interviewing a child, followed by moments of reflecting on that interaction, they recognize different types of questions and strategies that are more appropriate and effective in particular situations in the interaction.

Also in the context of teacher education, Almeida and Fernandes (2010) analyse the patterns of interaction and the kinds of questions spoken by prospective mathematics teachers. Almeida and Fernandes realized that they were extending the waiting time for the student's responses during their reflective practice. Once the questions were posed, then the prospective teachers realized they should wait for the students' answers. This is a particular learning consideration that can be used to engage in dialogue with students.

Chapin, O'Connor, and Anderson (2009, p. 6) state that classroom dialogue may provide "access to ideas, relationships among those ideas, strategies, procedures, facts, mathematical history, and more". In discussing these aspects, the students can be encouraged to "treat one another as equal partners in thinking, conjecturing, exploring, and sharing ideas". In fact, dialogue may provide different qualities in students learning when one compares to a classroom where the teacher monopolises the talk.

The literature above shows the importance of dialogue, and actions related to it, to learning mathematics. And, what about the learning of dialogue by prospective teachers? What reflections can be made concerning this topic?

2 About Dialogue in Mathematics Education

Since I began my teaching career, after my master's degree in mathematics education, thinking of dialogue as a means to promote learning has been a continual activity for me. From the master's degree, I remember the moments when I attended to undergraduates in their mathematics studies and tried to understand what they were thinking, by asking questions and listening to them. My teaching practice in secondary school and as a teacher educator brought new elements for my reflection on dialogue. For instance, I realized in the teaching practice courses that many prospective teachers agreed on the importance of listening to what the students have to say about what is being studied in mathematics and trying to understand them. However, in the prospective teachers' classes in schools, the vast majority did not put these actions into practice. What could be done in teacher education for prospective teachers to develop actions related to dialogue with their students? What could I say about the process of learning to be engaged in dialogue?

With this intention, I developed my doctoral research on the learning of dialogue by prospective mathematics teachers in the context of a teaching practice course. In that scenario, I had two main aims: to understand the process of learning how to be engaged in dialogue and to reflect on the concept of dialogue.

From my teacher practice and the literature presented in the previous section, I knew that it is difficult for prospective teachers and beginning teachers to listen to their students, consider different perspectives, pose questions, and, in general, establish an open interaction with them. I also knew the importance of dialogue for the learning. Considering that, I believed some actions in teacher education could be taken in favour of the learning of dialogue.

As an initial theoretical approach for data production for this research, the concept of dialogue was taken according to Alrø and Skovsmose (2004) in the context of critical mathematics education. Among other studies, their theoretical ideas about dialogue are based on Paulo Freire's political ideas about this concept. According to Freire (2005), dialogue is a meeting between people in order to think critically on events in the world and the possibility to change these. In this sense, dialogue is not a simple conversation. Dialogue is a means for people to name the world, to change the world. People constitute themselves in dialogue, in action and reflection, and not in silence. Therefore, dialogue is existential.

In dialogue, one cannot "deposit" her/his ideas in the other's head. Dialogue is not a war-like discussion in which one tries to force her/his ideas over the other's. On the contrary, dialogue is an act of creation. Being part of the dialogue is not a privilege of the few, but a right of people. There is no dialogue if there is no love (respect) for the world and people. It is not possible to be in a dialogue if one does not respect the other's contributions and ideas. There is no place in dialogue for self-centred people who do not need to learn with/from others. It cannot exist without humility. Dialogue is a meeting to create together, where there are no relations of dominance.

In the sense of that political perspective, Alrø and Skovsmose (2004) present the concept of dialogue related to learning in the context of critical mathematics education. The authors characterize empirically the dialogue between teacher and students and among students in terms of the acts that constitute the Inquiry Cooperation Model: getting in contact, locating, identifying, advocating, thinking aloud, reformulating, challenging, and evaluating. Considering the theoretical context of the dialogue—critical mathematics education—the authors highlight three aspects: making an inquiry, running a risk, and maintaining equality.

In the inquiry process, participants become involved cooperatively to discover something and to acquire knowledge and new experiences. An inquiry environment can be related to pure mathematics, semi-reality, or reality (Skovsmose, 2001). There is both an intention and an attitude of curiosity, which moves participants. They control the process together and are responsible for conducting the activities; this is a shared property. In this context, each participant can have a perspective, but it is necessary to strike a balance between advocating for and letting go of one's own thoughts in order to value the collective and to create and explore other perspectives.

Since these perspectives cannot be known beforehand, the directions a dialogue will take are unpredictable. When one wants to know what the other thinks, one can suspect something, but is not sure of how the other will respond. The verbal and nonverbal interventions "feed" and "give life" to the dialogue. Learning and inquiring in a dialogic scenario involve risk-taking and the challenge of experiencing new possibilities, which generates learning opportunities. A context characterized by unpredictability and learning possibilities is called a risk zone (Penteado, 2001). Venturing causes discomfort because one does not know whether the perspective will be well accepted. This doubt, which creates moments of tension, can be reversed into euphoria when, unexpectedly, a perspective is useful in the investigation process. A feeling of uncertainty in excess is not beneficial to the dialogue. Students can feel frustrated and lost and then give up. Alrø and Kristiansen state that the idea is not to remove the risk, but to promote momentary uncertainty (as cited in Alrø & Skovsmose, 2004).

When considering the knowledge that teachers and students have about a specific mathematical content, an asymmetric relationship between them is established: the teacher knows more than the students. What matters, however, when teacher and students are engaged in dialogue is a different kind of relationship, an egalitarian relationship. It does not mean that participants are equal, but that everyone is entitled to speak, and the differences and diversity in ways of acting and thinking are respected.

In making an inquiry, running risks, and maintaining equality, participants in the dialogue engage in more specific actions. The empirical characteristics of the dialogue refer to the set of actions, called dialogic acts, which help both the maintenance and the development of the dialogue. I outline below some of these dialogic acts: getting in contact, locating, identifying, and thinking aloud, reformulating, and challenging.

Getting in contact is a key element in a cooperative activity. Without this element, the cooperation process will not start. Often this contact is made by an invitation to explore something together. Maintaining this contact depends on the students' interest to remain involved in the activity. Therefore, it is not sufficient to establish an initial contact. One must also maintain it. For this interest to be renewed, the teacher must engage in active listening, which means "asking questions and giving non-verbal support while finding out what the other is getting at" (Alrø & Skovsmose, 2004, p. 62).

In collaborative work, finding out what the other thinks is critical. The act of *locating* is a dialogical process of expressing perspectives and making them visible in the interaction between participants. One tries to understand what the other thinks at a certain time during the activity or how they understand a specific problem. When someone suggests a way to solve the challenge, this perspective should be explored as a possibility for action. The dialogic act of *identifying* is related to the details of a perspective, when its peculiarities and implications to inquiry are expressed.

Amid the process of *locating* and *identifying* perspectives, students *advocate* in order to argue, defend, or reject ideas. Talking is a powerful tool for learning mathematics. The dialogical act of *thinking aloud* refers to the verbalization of reasoning to make a perspective public, and thus allow it to be investigated.

From the exposition of ideas by teacher and students, both can *reformulate* the other's thinking, in order to see whether the perspectives of each side are understood. When a teacher tries to understand what a student says and *reformulates* her/his ideas, for example, with questions like "did you mean that…?", the teacher shows her/his interest in listening. Thus, the student feels welcome to keep participating in the dialogue, in tune with the teacher.

Challenging means trying to push ideas in a new direction or questioning what is known. The challenge, however, cannot be done in just any way. It should suit the current conceptions of the student, and it cannot be too much or too little. Throughout the inquiry activity and, specifically, at the end of this process, it is important that both teacher and students *evaluate* the work as a whole and also the reasoning and the specific procedures used.

3 A Possible Context to the Learning of Dialogue

Taking inspiration from such theoretical concepts and my practice as teacher educator, a teaching practice course on mathematics was designed in order to promote a meeting between the prospective teachers and the concept of dialogue. The supervisor of the teaching practice course and I developed dialogue activities with reference to inquiry, planning, and reflection. The aim was to lead the prospective teachers to experience dialogue in landscapes of investigation, recognize themselves as people engaged in dialogue, analyse the elements of that dialogue and its benefits for the learning of mathematics, and imagine themselves as teachers engaged in dialogue with their students. As a fundamental part of the teaching practice course, after engagement in the dialogue activities, the prospective teachers planned and executed mathematics classes for the elementary level geared toward dialogue. This was done with my and the course teacher's supervision.

During the entire course, I assumed the position of practitioner-researcher as Jarvis (1999) considers the term. Jarvis calls the professional who develops research on her/his own practice a *practitioner researcher*. According to this author, the practitioner researcher knows what works in her/his practice, she/he feels comfortable in relation to the knowledge, skills, and attitudes from her/his practice, and knows what problems should be investigated. Throughout my practice, I often reflected on my actions to assess what needed to be maintained or modified. "Practice is both a site and an opportunity for learning, and reflective practice is a necessary approach to learning how to become an expert practitioner" (Jarvis, 1999, p. 70). Therefore, besides performing actions of my own practice, I reflected, in a systematic way, about them and their effects on the prospective teachers' decisions.

During the development of the dialogue activities and the prospective teachers' practice at schools, utterances and excerpts from conversations among them, between them and their students, and between them and me captured my attention. I constructed episodes related to those excerpts, which constituted the data of the research. On reflection, I could see that those episodes had a relationship with the kind of dialogue I was promoting for the prospective teachers to put the dialogue into action. Then, after preparing those episodes, I tried to write some statements about them from my own conceptions and suggested some interpretations and conclusions. It is in this sense that Creswell (2007) states that qualitative researchers interpret what they find, based on their own experiences and background. My initial reflections on the data drew on knowledge from my practice and my theoretical ideas about dialogue. These reflections formed my initial theories based on the data and which explained the way people participated in the research. As a first step, I needed to understand the meanings produced by the participants and reflect on them, following a naturalistic or interpretive approach (Denzin & Lincoln, 2006). After this I could move forward to more refined theory. At the end of the research, as a process of generalization, I reconsidered the empirical context and my reflections in order to develop statements about the learning of dialogue and the concept of dialogue itself.

In order to discuss the prospective teachers' attempts to put dialogue into action, I use Isabela's practice of dialogue to illustrate. Isabela is one of the prospective teachers who attended the course and participated in the research. I was present in Isabela's class at school. Soon after the particular lesson, which I will describe in the following section, Isabela and I met to reflect on and understand some events that had happened in the dialogue with her students, based on our memories about what had happen in the lesson. The transcripts below were produced after the lesson and my meeting with Isabela.

4 "Yes, I Have Heard What They Said": Isabela's Practice of Dialogue

Isabela was attending the final year of the mathematics teacher education program. She worked in the administrative sector of a company and was hoping, in this last teaching practice course, to see if she liked being a teacher or not. She had participated productively in the dialogue activities and in discussions in the teaching practice course, and she had accepted the invitation to put dialogue into action with her students at school.

Reflecting on Isabela's practice, as well as the other prospective teachers', would give evidence of what happens in the process of learning how to be engaged in dialogue, and it would help to discuss what is dialogue. The following episode (the -2 episode) concerns Isabela's attempt to put dialogue with her students into action. It is important to clarify that, in this transcript, my references to "students" do not include all the students in the class, but rather to a group of students. The students who talked is not the same every time "students" appears in the transcript. The same is true for my references to a "student". The transcripts below are a translation from Portuguese done by myself.

It is noteworthy to mention that I am using the word "dialogue" with the freedom to theorize about this form of interaction. Besides that, when I refer to the dialogue between Isabela and her students, I am referring to the interaction in the classroom which was planned and guided according to what we studied about dialogue in the teaching practice course.

In one lesson, Isabela developed the subject "complex numbers" with the students. I was present in the role of supervisor and researcher. Her practice of dialogue was full of questions that sought the participation of the students to suggest ideas, confirm mathematical facts, and provide specific answers. When Isabela expected these responses, and they were correct according to her way of thinking, she demonstrated that she knew how to handle this: Isabela accepted the answers, valued them, and incorporated them into her follow-up utterances. When she did not expect the students' answer, she reacted differently. I will present an example of such an unexpected answer in the following episode. Leading into this episode, Isabela and the students were representing some real numbers in the x-axis in Cartesian plane. Now they were beginning to consider complex numbers.

Isabela: And now? A complex number? How could we represent a complex number in the plane?

Student: Anyone?

Student: Square root of three.

Student: Square root of minus (pause) three.

Isabela: Square root… (pause) I am sorry. I did not understand you (looking towards the students who answered her). (pause). A complex number in the form $a + bi$? I have $1 + 2i$. Where could I plot this number? (pause) Does anyone have any idea? (pause) Does anyone have any idea how I could plot it?

Student: Minus two.

(pause)

Isabela: What did we also study? A complex number $1 + 2i$ is composed of two parts, isn't it? A real part and another one which is…?

Student: Imaginary.

Isabela: Imaginary. What is our real part?

Students: It is one.

Isabela: It is one. Okay, then one is here (she plots it on the horizontal axis). What is our imaginary part?

Students: $2i$.

Isabela: $2i$. Where could I plot $2i$? (Note: Isabela states that the imaginary part of the number $1 + 2i$ is $2i$, which is not true according to the theory of complex numbers).

Students: Minus two.

(pause)

Isabela: Plus $2i$ (writing on the board). What does i really mean? (pause) Who is our i? Did everybody forget?

Student: Minus one.

Isabela: Minus one? (Isabela looks surprised) Who is our imaginary unit? (smiling) Who did we set as our imaginary unit?

Students: Minus one.

Student: Square root of minus one.

Isabela: Ahhh… square root of minus one.

Student: But there is minus one in here!

Isabela: Okay (laughing). But now, where did you think I could plot?

Isabela did not have any answer from the students to her last question in the above episode. Immediately following this moment, she introduced the Argand-Gauss plane, showing the real axis and the imaginary one. When she asked again how to plot $1 + 2i$ in the plane, the students answered "1" for the real axis and "-2" for the imaginary one. Once again "-2" came as an answer. The dialogue had been diverted by a question from a student who had missed the previous lessons and, when Isabela resumed, she told the students how the complex numbers are represented in the Argand-Gauss plane.

5 Data About Data: Isabela's Point of View About Her Own Practice

For someone who reads the episode above, without having been present at the lesson or talked with Isabela about it, it may seem she had ignored the students' answers, especially when they responded "-2". An exception could be considered when she said "I am sorry. I did not understand you". Even though I was present there and saw Isabela's attempt to put dialogue into action, even my impressions are an external look at her practice.

Thus, I ask, what does Isabela say about what she thought and did? How does she see her practice of dialogue? The data represented in the episode above are not sufficient to answer these questions. Knowing the answers to these questions would be essential to understand Isabela's practice of dialogue and to be able to say something about the process of learning to be engaged in dialogue. Therefore, new data was produced to build on the first set of data. This is data about data, that is, a reflection and clarification of the initial data (the -2 episode).

Soon after the lesson described above, Isabela and I met to reflect and understand what had happened. This reflective dialogue began when I asked Isabela if she had heard her students when they answered "-2" for the representation of the complex number "$1 + 2i$" in the plane. Then, she said:

> I heard -2 and then I do not know if I asked why -2, but I wrote (on the blackboard) "$1 + 2$ …" and asked, "Who was i?" In this moment I tried to answer about that -2. I think this -2 came because of the -1, $\sqrt{-1}$. I think that was the reason, and then I tried to rewrite there "okay, but what is $1 + 2i$?". And then I rewrote the $\sqrt{-1}$ (in the place of i) to show that it was not -1 times 2, but rather $\sqrt{-1}$ times 2, i times 2. I tried to show in this moment.

Isabela heard the answer "-2" and took it in consideration. Then, we talked about the importance of welcoming the students' answers in dialogue. Isabela said she did not ask for a reason for the answer "-2" in order to know if her justification matched with the students'.

Raquel: But this is a question of practicing and training. A good first step was what you did, that is, trying to imagine what the student thought.
Isabela: Yes, I did that (Isabela looked to be happy with that).
Raquel: Perfect (we both laughed)! From now, we can have this justification explicit to the students, because maybe other ones are thinking on -2 and did not understand why it is not -2.
Isabela: Hmmm. Because I wrote that (the substitution of i by $\sqrt{-1}$) in the little corner of the blackboard, but it was not clear.
Raquel: This justification was clear just for you.
Isabela: Yeah.

Somehow, Isabela had her justification for the answer "-2" written in the corner of the blackboard to some students, and we agreed that it had not become clear for the whole class.

In that moment of reflection and supervision, Isabela explained her point of view about the -2 episode. She had heard the students' answers. Her pauses in the episode became clear: Isabela remained silent, took the unexpected utterances into consideration and reflected on them, imagining what the students might have been thinking in order to say what they had said. Then, she formulated a possible justification for the students having thought in that way and, somehow, made it explicit to some of them (in the corner of the blackboard).

6 About Learning of Dialogue

Starting from what was done by Isabela in her class, we highlighted what she did not do. This led us to initiate a process of pedagogical imagination (Skovsmose, 2009) in order to get a glimpse of the possibilities for what could be done in her next classes, considering other aspects of dialogue. Imagining what could be is not a neutral task. One can imagine many possibilities for action. However, they are strongly influenced by the intentions of the people involved in the process, and these intentions are created from the way the people understand the actual situation.

This creative and reflective process is valuable to teacher education, as it reveals possibilities for teaching practice. "The existence of an imagination that describes alternatives to an actual situation makes a difference. By this imagination, experienced necessities could be reduced to contingencies—they *could* be different" (Skovsmose, 2014, p. 124, emphasis in original). In this sense, the situations presented to Isabela in her practice could be rethought and not accepted as natural. To consider, with Isabela, what she did not do and what still could be done has provided an opportunity to think about possibilities for her practice of dialogue. This describes the cooperative development of a pedagogical imagination by a prospective teacher and a supervisor.

In addition to imagining what the student thinks, in future classroom dialogues the teacher can ask students to explain to the whole class how they thought when they say something unexpected. Furthermore, the teacher can ask questions to better understand a student's way of thinking that generated unexpected answers. Isabela advanced in this way in her following classes.

Isabela's practice of dialogue and the moment of supervision brought to mind some ideas from my own practice as a teacher educator and my master's degree experiences. Considering the interaction with her, and inspired by the theory I have brought to the context, I elected some essential elements of the dialogue to elaborate below: active listening (Alrø & Skovsmose, 2004), estrangement (Lins, 2004), and decentring (Oliveira, 2012). Isabela's attempt to put the dialogue into action with her students can be read (for me) as an effort to exercise these elements.

Estrangement and decentring are notions based on the work of Lins on the context of interaction. Lins (1999) characterizes interaction in terms of meaning production. This excerpt of Lins' work could be read while imagining that a teacher is talking to a student:

> I do not know how you are; but I need to do. I also do not know where you are (I just know you are in some place); I need to know where you are and so I can go there and talk to you and we can understand each other, and negotiate a project in which I would like your perspective of going to new places to be present. (Lins, 1999, p. 85)

Lins does not consider the expression "where you are" as a physical place or a stage of cognitive development, but rather as legitimacy of meanings for a person.

With this approach to meaning-making, active listening is not just listening to what the other says, but rather "asking questions and giving non-verbal support while finding out what the other is getting at" (Alrø & Skovsmose, 2004, p. 62), as I have described above. By listening actively, when one notices a difference among ways of thinking, such as the answer expected by Isabela and what her students have enunciated, an estrangement may happen. Something that is not expected and natural for a person comes up. In this process of estrangement, "what really matters is that there is a part for whom the thing is natural—be it strange or not—and another one for whom this thing could not be said" (Lins, 2004, p. 116, my translation).

When estrangement is not taken into consideration, dialogue is less likely to last or even to happen. When the teacher chooses to face estrangement, she/he is interested in seeking to understand the (cognitive) place from where the student talks. It is a move of being out of her/his place to go to where the student is. This is described as decentring, which is described by Oliveira (2012) as the "effort of becoming sensible to the other's estrangement, and understanding what the other says" (p. 207, my translation). In Isabela's practice of dialogue, when Isabela experienced estrangement, she tried to understand what her students were meaning by "-2". This effort is characterized as a *changing of the centre*.

When the teacher found herself/himself in this place ("I understand how you are thinking"), she/he could begin to work with this new knowledge, understand the difference of meanings, incorporate the student's thoughts into her/his talk, share with the whole class the new idea, and clarify that different thoughts are at work in the situation.

The move of going to where the student is constituted by active listening, that is, an attentive listening to what the other says, an effort to understand him/her, a non-verbal support, and inquiry questioning about the student's thought.

7 Dialogue as a Move Towards the Other

Here I propose a new interpretation of dialogue in the context of mathematics education. This comes from my reflection on the practices of dialogue among the prospective teachers who have participated of the research, for example, the case of Isabela. It is underpinned by the theoretical inspirations I described above, especially from the concept of dialogue by Alrø and Skovsmose, and the conception of interaction by Lins.

Dialogue is a form of interaction between teacher and students engaged in a learning activity in which talk and active listening are shared, the ideas are discussed, and an understanding of what the other says is fundamental (Milani, 2015). Dialogue is something that is done with the other. In the educational context, assuming a dialogical stance means that the teacher and the students share the talk, that is, the speech is not monopolized by one of the parts. This is a political stance. Considering equality in dialogue, everyone has the right to express her/his own perspectives. Being engaged in dialogue with the other means listening to him/her, asking the other questions, and being interested in what the other says.

Because dialogue is to be with the other, it is also a move towards the other. The interpersonal aspect of interaction is on the basis of the dialogue. When the teacher and students are engaged in dialogue, they are influenced by each other. There is an encounter of meanings. Meanings are shared through language in the interaction. From this encounter, the teacher can learn how the students produce meanings for the objects involved in an activity and learn about the way that activity is developed (what works and what does not). Similarly, the students can learn new meanings and ways of thinking from paying attention to the teacher's and classmates' meanings. Therefore, the teacher learns from the student and the student learns from the teacher in the process of dialogue, as Freire states.

In the discussion of ideas in a learning activity, whether there is controversy or not, it is inconceivable to impose one's perspective on the other. Dialogue is not a competition; there is no winner or loser. What exists is a move toward each other, that is, an attempt to understand what the other says.

8 Final Remarks

Isabela's practice of dialogue and the moment of pedagogical imagination enabled me to reflect on the process of learning to be engaged in dialogue and, especially, on the concept of dialogue as a move to go to where the other is. In the context of teacher education, some actions may be part of the process of prospective teachers learning to engage in dialogue, for example: to experience dialogue in investigative activities, to recognize oneself as someone engaged in dialogue, to be involved in pedagogical imagination to imagine oneself as a teacher engaged in dialogue; to create imaginary dialogues that anticipate the utterances and actions of people involved in the situation, to change closed patterns of communication into more dialogical interactions, and, during the teaching process with supervision, to constantly create new possibilities to have a context of dialogue in which the essential elements are put into action.

Considering my practice as teacher and teacher educator, Isabela's practice of dialogue, and the literature presented here, I elected three essential elements of dialogue: active listening, estrangement and decentring. These elements fit Isabela's practice well. When I looked at Isabela's practice it was relatively easy to recognize and reflect on these elements.

This draws my attention to the other data—the dialogue practices of the other prospective teachers who participated in the research. When I use the lens of these three elements to look at the more complicated data from those interactions, what do I see? Are the essential elements in those practices? Though I have not done full analysis of these interactions, I can already say that this last question is not a yes-or-no question. It is a reflective one. It seems that there are aspects that can impede and trigger dialogue. When I use the lens developed in my interaction with Isabela to understand other interactions, it is a way of problematizing the essential elements of dialogue. When I use the three elements in other interactions, they are not as clear. However, the lens represents a step forward in understanding dialogue.

To close, I would like to stress that the choice of dialogue in mathematics education is not impartial. My readings on this concept in the context of critical mathematics education influenced me to consider equality in interpersonal relations as an important aspect of my thoughts on dialogue. I emphasize my approach by asking opposing questions: Why not focus on talk monopolized by the teacher? Why would a teacher be engaged in dialogue with students? Before taking a pedagogical stance, it is worth recognizing that the choice of dialogue shows a political stance of the teacher. It is in the dialogue that one moves toward the other in order to understand what she/he says. But why is it important to know what the other says and thinks? I am not alone in the world. I am and I act *with* the other. In the context of mathematics class as a teacher, I am *with* the students. In the context of mathematics teacher education as a supervisor, I am *with* the prospective teachers. I have great interest in what the other says, because it is in that way I can go with him/her to new places. It is not about giving voice to the students, expressing power, or domination: "Speak, I let you speak". No. Active listening is not an exercise of power. Rather, it is an exercise of freedom: "Speak, I listen to you".

What I want to emphasize is the interpersonal aspect of dialogue as I understand it: the dependence of knowing what the other says in order to have the dialogue happening and lasting, the realization that one's action triggers the other's, and active listening that includes estrangement and decentring as part of the move to go to where the other is.

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Part IV
Language Diversity Research

Teacher Knowledge and Teaching Practices in Linguistically Diverse Classrooms

M. Alejandra Sorto, Aaron T. Wilson and Alexander White

Abstract This study sought to empirically estimate teachers' mathematical knowledge and knowledge of teaching linguistically diverse learners, and to shed light on the links that may exist between mathematical and teaching knowledge and classroom practices. Correlational analysis showed that there were strong associations between teachers' mathematical knowledge and knowledge of teaching linguistically diverse learners. These in turn were strongly associated with rich mathematics and attention to students as learners during instruction.

1 Motivation and Goal

Currently in the United States, gaps in mathematics achievement scores exist between linguistically diverse students and their majority counterparts. Nevertheless, recent mathematics achievement results in the state of Texas show that some districts outperform others despite having large percentages of language-minority and socio-economically disadvantaged students. Therefore, the main goal of this study was to provide insights and explanations for this phenomenon from the teacher capacity perspective by investigating how teachers' mathematical knowledge, knowledge of students as linguistically diverse learners, and quality of instruction contribute to students' achievement gains. In particular, we explored the relationship between teachers' knowledge and their teaching practices in linguistically diverse middle school classrooms.

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© Springer International Publishing AG, part of Springer Nature 2018

J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_16

2 Literature Background

Studies in US bilingual mathematics classrooms have focused on different research areas. One is the relationship of learning mathematics to cultural issues such as students' households, parents, and communities (Civil, 2007). This perspective does not examine the practice of teaching but instead explores how the out-of-school world of the learners relates to the classroom. Another deals with mathematics and 'social activism' (Gutstein, Lipman, Hernandez, & de los Reyes, 1997), equity, and empowering minority students (Celedón-Pattichis, 2004). For example, Gutstein and others looked at the teachers' and principals' beliefs and ideologies in relation to integrating mathematics with the children's culture, while Celedón-Pattichis investigated the role of parents, educators, and administrators together with school policies that place English language learners (ELLs) in mathematics upon immigration to US schools. Finally, a third category, more pertinent to this study, deals with mathematics practices and language in or out of the classroom (Civil, 2007; Khisty & Chval, 2002; Moschkovich, 2007). Part of this body of research focuses on how instruction can provide opportunities for students to talk about mathematics. Civil (2007) explored mathematics learning in different settings like regular classrooms, after-school mathematics clubs, out-of-school activities, and with parents. Moschkovich (2002) looked at the use of two languages (code-switching) and gestures during mathematical conversations in classrooms. She analyzed classroom discussions to show how students use two languages as resources for mathematical communication, contrary to the view that code-switching indicates deficiency. Khisty and colleagues (2002) analyzed the role of the teacher during mathematical discussions. They argued that even students proficient in English can experience difficulties in communicating during mathematics discussions because of differences between proficiency in 'social language' and 'academic language'. Furthermore, they concluded that the teacher plays a significant role by using her/his own academic talk as a model and support for students' emerging mathematical discourse. Despite this body of research on the teaching and learning of mathematics for ELLs, there is a lack of detailed understanding of the relationship between teacher knowledge and classroom instruction in linguistically diverse classrooms. Hence, the research question addressed in this study was:

> To what extent are teachers' knowledge measures and the mathematical quality of instruction in linguistically diverse classrooms associated?

3 Methodology

3.1 Measures

The study utilized established measures for teacher knowledge and quality of mathematics instruction with complementary measures to capture the special knowledge and instruction in linguistically diverse classrooms. The primary variables in this study were the teachers' mathematical knowledge for teaching performance, pedagogical content knowledge for teaching mathematics to English Language Learners, and quality of mathematics instruction.

3.2 Mathematical Knowledge for Teaching (MKT) Instrument

The Learning Mathematics for Teaching (LMT) project at the University of Michigan developed the MKT instrument. This is a rigorously developed and validated instrument designed to measure "…not only the mathematical knowledge common to individuals working in diverse professions, but also the subject matter knowledge that supports that teaching, for example, why and how specific mathematical procedures work, how best to define a mathematical term for a particular grade level, and types of errors students are likely to make with particular content" (Hill, Ball, & Schilling, 2008, p. 431). Mathematical Knowledge for Teaching (MKT) includes both the domains traditionally conceived as pedagogical content knowledge (knowledge of students, knowledge of content and teaching), as well as two types of subject matter knowledge itself: common content knowledge and specialized content knowledge. In addition, the instrument comes with a set of background questions about teacher education and professional development opportunities. These questions were modified to fit the population under investigation. There were two main reasons for using this instrument for this study. One is because it is the most rigorously developed and validated instrument that measures teachers' knowledge in this country. The second reason is because it has been linked to student achievement gains (Hill, Rowan, & Ball, 2005) and to the quality of mathematics instruction (Hill et al. 2008).

However, since that instrument is not sensitive to the complexities of teaching mathematics in the linguistically and culturally diverse classrooms in which this study transpired, we supplemented the MKT measure by developing an original instrument to measure knowledge for teaching mathematics to linguistically diverse learners, discussed below.

3.3 Pedagogical Content Knowledge for Teaching Mathematics to ELLs (PCK-MELL) Instrument

In considering the teachers' knowledge that might inform their instructional decisions (and impact student achievement), we hypothesized that teachers may possess a special kind of practice-based pedagogical content knowledge about teaching ELLs that they draw upon when teaching these students. Since the MKT instrument was not sensitive to capturing teachers' knowledge of linguistic or cultural issues that might arise in the classroom, especially any issues related to teaching ELLs, we developed the PCK-MELL instrument for the purpose of measuring this additional component of teachers' knowledge. In doing so we followed standards for the development of psychometric instruments such as those given by DeVellis (2012). We defined the PCK-MELL construct based upon our reading of the literature which has shown that ELLs often encounter a number of specific obstacles in the mathematics classroom, such as linguistic complexity (Martiniello, 2009) or the multiple meanings for English words (Lager, 2006). Alternatively, researchers have also shown how ELLs' first language, their cultural knowledge and prior academic knowledge provides resources for learning mathematics (Gutiérrez, 2002; Gutstein et al., 1997; Moschkovich, 2002). Additionally, a large number of instructional strategies for teaching ELLs have been derived from the research and offered in the practitioner literature (for instance, see Coggins, Kravin, Coates, & Carroll, 2007). Based upon these findings we conceptualized the PCK-MELL construct as a subset within Hill et al. (2008) larger MKT construct. Figure 1 illustrates this theoretical framing.

Using the conception of PCK-MELL given in Fig. 1 (and an associated test blueprint based upon this theoretical framing), a large number of items were developed and the final 32-item instrument was piloted with pre-service mathematics teachers and practicing middle school mathematics teachers (Wilson, 2016).

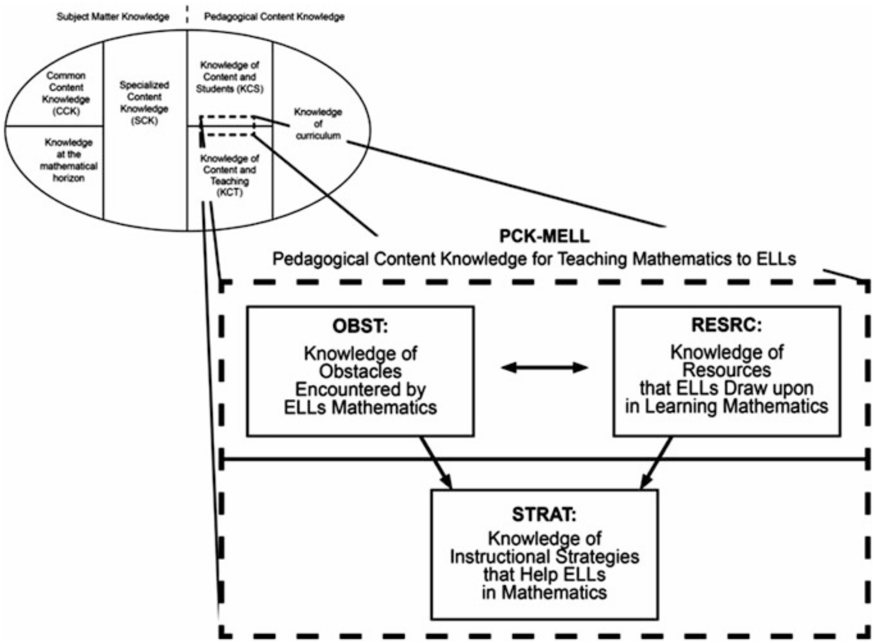

Fig. 1 Pedagogical content knowledge for teaching mathematics to ELLs as a subset of MKT

The resulting instrument, which was used in this study, has items intended to capture teachers' knowledge of the linguistic and mathematical obstacles faced by ELLs, of the linguistic and cultural resources that these students invoke in doing mathematics, and also of teachers' knowledge of strategies for teaching ELLs.

3.4 Mathematical Quality of Instruction Observational Protocol

We captured the classroom practices with the 4-point version of the Mathematical Quality of Instruction (MQI) protocol (Hill et al., 2008). This instrument measures several dimensions that characterize the rigor and richness of the mathematics in a lesson, including the presence or absence of mathematical errors, mathematical explanations and justifications, mathematical representations, and related observables. More specifically the protocol consists of 30 codes grouped into five domains. Codes for the first four domains (20) are recorded in segments of 7.5 min and the remaining 10 codes are recorded as a whole lesson. One of the whole-lesson codes is an overall MQI code. There are four possible scores for segment codes: Not Present, Low, Medium, and High (0, 1, 2, and 3) and five possible scores for whole-class codes ranging from 'Not at all true of this lesson' (1) to 'Very true of this lesson' (5). Table 1 lists main domains of the protocol.

Although one of the codes within the section of Richness of the Mathematics includes mathematical language which captures how fluently the teacher (and students) use mathematical language and whether the teacher support students' use of mathematical language (Hill et al., 2008), this does not capture other elements related to the use of mathematical language while using and learning multiple non-mathematical languages. Hence, the MQI was augmented by a new domain called Quality of Linguistically Diverse Teaching which is described next.

Table 1 Main sections of MQI and corresponding number of codes

Main domains of MQI	Number of codes
Richness of the mathematics	7
Working with students and mathematics	3
Errors and imprecisions	4
Common core aligned student practices	6
Whole-lesson codes	10
Total	30

3.5 Quality of Linguistically Diverse Teaching

The new domain was developed by the authors based on previous research (Chval & Chávez, 2011) and validated with a focus group of teachers and a small set of existing videos from linguistically diverse classrooms (Sorto, Mejía Colindres, & Wilson, 2014). It consists of six segment codes which are presented in Table 2. Consistent with segment codes from the other domains of MQI, the possible scores for each code ranges from Not Present (0) to High (3).

Table 2 Quality of Linguistically Diverse Teaching codes

Segment codes	Description
Connections of mathematics with students' life experiences and prior knowledge	This code captures instances by which the teacher activates students' prior mathematical knowledge by explicitly referencing skills learned in a previous lesson or grade. This code includes references by the teacher to mathematics found in daily life by students such as money and shopping
Connections of mathematics with language	Teachers or students connect language (words) with mathematical representations such as pictures, tables, graphs, and mathematical symbols. This code captures the extent to which the teacher reinforces a mathematical representation with its meaning
Meaning and multiple meanings of words	Teacher or students communicate meaning by using synonyms, gestures, drawings, cognates, or translations to students' first language that supports learning. This code includes reading strategies meant to increase comprehension. Meaning that occurs between students that is correct can adjust the score upward
Use of visual aids or support	Concrete objects, videos, and illustrations are used by the teacher or students in classroom conversations. Concrete objects may include times tables, formula charts, protractors, 2D models, or dynamic foldables
Record of written essential ideas and concepts on board	Teacher displays a written record of the lesson's essential ideas and concepts without erasing so students can refer to them throughout the lesson. The score may be adjusted downward if some important/essential information is never recorded
Discussion of students' mathematical writing	Teacher use students' written work as an instructional tool and point of discussion

3.6 Sample and Data Collection

The participants of the study were 34 middle grade mathematic teachers representing all of the 11 middle schools in a large district in south Texas with even distribution among grades 6, 7, and 8. About two thirds of the teachers self-identified as being either advanced or advanced high in Spanish language proficiency. The average teaching experience was 9.5 years and the majority of them (75%) taught in classrooms with at least half of students classified by the school district as ELLs. About 99% of the students in the district were from low income homes (in the U.S. this is determined because the students qualify for free/reduced lunch). Table 3 presents more detailed teacher characteristics and frequencies with respect to their levels of certification and education background.

Teachers responded to the two surveys during a professional development session in summer 2013. The MKT instrument was administered to 34 teachers and the PCK-MELL instrument was administered to 32 teachers. The following school year (2013–2014), teachers were videotaped three times (beginning, middle, and end of the school year). They were asked to choose the classroom with the highest percentage of ELLs for videotaping. A total of 98 lessons were videotaped; thirty-one teachers were videotaped three times, two teachers were videotaped twice and one teacher was videotaped once.

4 Data Analysis and Results

4.1 Scoring Quality of Instruction and MKT

Four coders completed an online MQI training program (approximately 16 h). Videotaped lessons were broken into 7.5-minute segments which were then coded independently by two trained coders using the 30 codes designed to represent the elements of mathematical quality (outlined in Table 1). Since the desired threshold percent agreement of 80% was never reached, the pairs of coders had to meet and reconcile conflicting codes afterwards. Similarly, coders were trained for the augmented section of MQI, the Quality of Linguistically Diverse Teaching, by becoming familiar with the descriptions of the six segment codes and scoring a selected number of lessons. The interrater reliability for this newly developed section was measured by Cronbach's Alphas in three different pairs of coders and they are presented in Table 4.

Video scores for each teacher in each element and overall lesson scores were computed by averaging all three lesson scores. The MKT pencil-and-paper responses of the 34 teachers were compared with a larger national sample using a

Table 3 Teacher characteristics and background

Teacher characteristic	Distribution	Frequency
Years teaching (years)	0–4	6
	5–10	9
	11–20	10
	20+	4
Largest percent of ELL in any class (%)	0–19	4
	20–39	15
	40–59	6
	60–79	2
	80–100	5
Grades levels certified	Elementary	7
	Mathematics 4–8	17
	Mathematics 8–12	5
Special certifications	ESL	5
	Bilingual Education	3
	Special Education	4
Undergraduate courses taken		
Mathematics	No course	4
	One course	3
	Two courses	2
	3+ courses	25
Methods (pedagogy)	No course	10
	One course	5
	Two courses	6
	3+ courses	12
Professional development type related to teaching ELLs	Shelter instruction	22
	English Language Proficiency Standards (ELPS) Academy	13
	Language Proficiency Assessment Committee (LPAC)	5
	College course	8
Professional development in mathematics in the past year	None	3
	Less than 6 h	4
	6–15 h	17
	16–35 h	4
	More than 35 h	6

two-parameter IRT (Item Response Theory) model. Scale scores for each teacher were computed using parameters estimates for each item derived from the results of the national sample. The distribution of scores for our sample was very similar to the scores for the national sample of teachers.

Table 4 Interrater reliability for the Quality of Linguistically Diverse Teaching section

Pair of coders	Number of codes	Cronbach's alpha
Coder 1 and coder 2	665	0.855
Coder 1 and coder 3	2429	0.801
Coder 1 and coder 4	1099	0.816

4.2 Data Analysis and Results

The first analysis corresponds to the descriptive nature of the quality of instruction. Table 5 presents the mean and standard deviations of the newly developed Quality of Linguistically Diverse Teaching Scales. The first six scales in Table 5 correspond to teaching elements known to be effective in linguistically diverse settings (Chval & Chávez, 2011). From the table, we can see that teachers in the study were most attentive to writing essential ideas and concepts on the board. The mean for this scale was at least a full point higher than the other five elements. The two elements related to vocabulary, connecting mathematics with language and meaning of words, were observed at a moderate level. On the other hand, the element of connecting mathematics with students' life experience was rarely observed. After coding for the six elements, researchers assigned a separate score from 0 to 3 to capture the overall quality of the lesson. This holistic score was then averaged across the lessons to determine the Overall Quality of Linguistically Diverse Teaching. The teaching observed in this study was determined to be of moderate quality with a mean of 1.61 out of 3.

Table 6 presents the summary statistics for each segment of the MQI protocol as well as the holistic whole lesson code. For comparison purposes, the Linguistically Diverse Teaching scale is also included. The low mean for the Errors and Imprecision domain (0.34) shows that the teachers in the study rarely made mathematical errors. In this study, the scores above 0 were usually due to imprecisions of language rather than to errors in computations or derivations. Meanwhile, the other 3 domains from the MQI protocol (Richness, Working with Students, and

Table 5 Mean and standard deviation of Linguistically Diverse Teaching scales

Linguistically diverse teaching scales	Mean (0–3)	Standard deviation
Connections of mathematics with students' life experiences	0.28	0.27
Connections of mathematics with language	1.32	0.41
Meaning and multiple meanings of words	1.16	0.40
Use of visual aids or support	0.70	0.65
Record of written essential ideas/concepts on the board	2.32	0.39
Discussion of students' mathematical writing	0.97	0.51
Overall quality of linguistically diverse teaching	1.61	0.34

Table 6 Mean and standard deviation of teaching quality domains

Observational teaching quality domains	Scale	Mean	Standard deviation
Overall richness of the mathematics	0–3	1.33	0.26
Overall working with students and mathematics	0–3	1.38	0.32
Overall errors and imprecision	0–3	0.34	0.26
Overall common core aligned student practices	0–3	1.26	0.26
Overall Quality of Linguistically Diverse Teaching	0–3	1.61	0.34
Overall MQI	1–5	3.13	0.50

Common Core Aligned Practices) have means between 1.26 and 1.38. We thus conclude that the quality of teaching with respect to those aspects was between a low and middle level. The Linguistically Diverse Teaching scores were higher on average. From Table 5, we see that this is mainly due to teachers' cognizance of the importance of writing essential ideas on the board. The overall MQI score average of 3.13 is consistent with a mid-range quality of instruction.

In order to relate the quality of instruction to teachers' knowledge, the researchers used the well-known MKT measure and the recently developed PCK-MELL instrument. The results for these scales, as well as, two subscales of the PCK-MELL instrument are presented in Table 7.

When scores on the PCK-MELL instrument for the teachers that participated in this study are compared with the prior calibration sample (pilot-study), which included both mathematics teachers that had experience teaching ELLs and others that had very little or no experience with ELLs, there is evidence that the teachers that participated in this study, all of whom worked with ELLs, had slightly greater scores. The distribution of scores on the entire instrument was very similar to the calibration sample. However, on the two subscales of the instrument that measured, respectively, teachers' knowledge of linguistic issues (obstacles and resources) that ELLs encounter and of strategies for teaching ELLs, the mean scores for teachers in this study was slightly greater than for the calibration sample. This may not be surprising, since all of these teachers worked with ELLs. Furthermore, the standard deviation of scores for all measures of PCK-MELL for this study was slightly less than for the calibration sample which may also confirm that teachers in this study had similar experiences with ELLs and similar ways of thinking about teaching these students mathematics.

Table 7 Mean and standard deviation of knowledge for teaching measures

Teacher knowledge measures	Scale	Mean	Standard deviation
Mathematical knowledge for teaching (MKT)	0–92	52.68	11.11
PCK for teaching mathematics to ELLs (PCK-MELL)	0–32	18.87	3.73
Knowledge of linguistic issues	0–11	5.81	1.56
Knowledge of ELL teaching strategies	0–10	7.03	1.40

The next analysis used Pearson's correlation to measure the association between the teacher knowledge measures. Figure 2 shows the observed relationship between MKT and mathematical richness of the lessons. The correlation was positive and significant (0.482, p = 0.0213), suggesting that teachers with better developed practiced-based content knowledge may also better understand the mathematical strengths and weaknesses of their linguistically diverse learners. Table 8 presents the correlation between the overall elements of mathematical quality of instruction and the MKT measure. All of the associations were positive, except for the errors and impressions, which indicates that teachers who tend to make fewer mistakes in class have higher levels of mathematical knowledge for teaching. The same trend was observed in the measure of knowledge for teaching diverse learners. In both

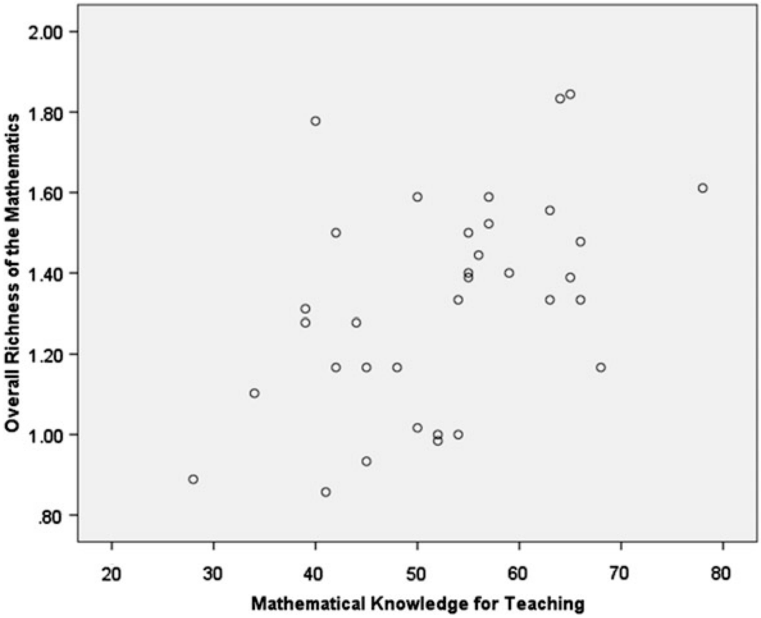

Fig. 2 Association between MKT and richness of mathematics

Table 8 Correlation of observational teaching quality elements with MKT

Observational teaching quality elements	Correlation with MKT
Richness of the mathematics	0.482 (**)
Working with students and mathematics	0.359 (*)
Errors and imprecisions	−0.288
Common core aligned student practices	0.256
Overall mathematical quality of instruction (MQI)	0.285
Overall quality of linguistically diverse teaching	0.311 (+)

+ Significant at the 0.1 level. * Significant at the 0.05 level. ** Significant at the 0.01 level cases this association was not significant. Two elements of Richness of Mathematics and Working with Students and Mathematics had significant positive associations with MKT; the Overall Quality of Linguistically Diverse Teaching was less significant ($p = 0.078$). The element of Richness of Mathematics was also significantly associated with knowledge of mathematics for ELLs (PCK-MELL). There was no evidence of any association between classroom observational elements (MQI and Quality of Linguistically Diverse Teaching codes) with the PCK-MELL measures. It was expected that this teacher knowledge measure would be correlated with the new dimension measuring the quality of linguistically diverse teaching. However, this was not the case and, unlike the alignment of the MQI codes with the MKT instrument, the PCK-MELL instrument itself also had few items that were aligned with elements captured by the observational instrument.

5 Conclusions

The present descriptive and correlational study gave results that demonstrate strong links between general mathematical knowledge for teaching and the teaching of mathematics to linguistically diverse learners. These measures in turn have strong links with the presence of rich mathematics presented to students and the teachers' ability to attend to and correct students' work during instruction. Links are moderate between teachers' knowledge and the ability to implement strategies that support the learning of mathematics for linguistically diverse students. Subsequent analyses will shed light on the extent to which these variables explain or contribute to students' learning gains.

These findings suggest that teachers with greater MKT, i.e., with a keener understanding of the ways in which to represent mathematics to students and of common student misconceptions for example, may also have an advantage in making instructional decisions that can support linguistically diverse students. While we may expect more knowledgeable teachers to make better decisions for their students, if we want to provide equitable access to learning *for all* students, including emergent bilingual students and those from different linguistic and cultural backgrounds, we *need* teachers to make better instructional decisions. These findings, although not conclusive, suggest elevating teachers' MKT as a possible direction for improving access and educational opportunities for language minority students. Similarly, findings like these have implications for mathematics educators who prepare future mathematics teachers to teach in linguistically diverse classrooms and for educational administrators who place the most suitable teachers in such classrooms. Moreover, given the connection that this study found between teachers' knowledge and the presence of rich mathematics in classrooms with teachers' ability to make effective use of strategies that support language learners and to respond to students' learning needs, further research into the specific kinds of strategies and teaching moves that best support these students as well as into the knowledge and experiences that best facilitate teachers' acquisition and implementation of these strategies is warranted.

Acknowledgements This research was supported by the National Science Foundation grant DRL-1055067. The authors wish to thank the administrators, teachers, and students who allowed us into their classrooms.

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Podcasts in Second Language Mathematics Teaching as an Instrument for Measuring Teachers' Language Awareness

Christine Bescherer and Pelagia Papadopoulou-Tzaki

Abstract Mathematics teacher students work for one year in local high schools in a special program for mathematics learning focusing on schoolchildren with German as a secondary language. To measure the change of language awareness, teacher students create audio podcasts on a mathematical topic at the beginning of their fostering courses. Further, in semi-structured interviews they give insights on their ideas about using mathematics podcasts in classrooms. After one year working as remedial teachers with schoolchildren, the mathematics teacher students review and revise their 'old' podcast. Afterwards structured interviews with each teacher student and a qualitative analysis are used to diagnose the level of language awareness. The podcasts together with the interviews prove to be useful instruments to reflect on the use of language in mathematics teaching.

1 Introduction

The project FörBis (Fostering Cognitive Academic German Language Proficiency - Förderung der Bildungssprache Deutsch) funded by the Mercator Institute, Cologne, aims to develop materials and methods for fostering German as a secondary language (GSL) in mathematics classrooms. The explorative Ph.D. study of the second author focuses on remedial afternoon classes for schoolchildren in secondary schools. These classes aim to improve the mathematical skills and competencies as well as the German language skills and competencies of the schoolchildren. These classes are taught by advanced teacher students for a whole school year in schools around Ludwigsburg. The materials and methods used, including worksheets, schemes for text analysis of word problems, models for solving word problems, and glossaries are provided by the second author.

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J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_17

The overall research question in the Ph.D. study was whether there would be a (positive) impact of the whole setting on the level of the schoolchildren as well as the level of the teacher students. To document this impact several research methods were used, specifically standardized testing for mathematics and German language skills and informal tests regarding word problem solving strategies for the schoolchildren, in addition to semi-structured interviews about the usability of the material and teaching methods as well as language awareness of the teacher students.

In this paper we focus on the two questions

- Is there a change/increase of mathematics teacher language awareness (MTLA) after one year of teaching remedial classes and attending a seminar?
- Are podcasts effective for documenting or even measuring a change in the MTLA?

An overview of the aims, the teaching scenarios, and the research questions of the whole Ph.D. project is given in Fig. 1.

The schoolchildren were sent to the remedial classes by their regular teachers based on their diagnosed need for support in German language and mathematics. For most of these selected schoolchildren, German is not their mother tongue.

The teacher students who took part in this study were majoring as mathematics teachers and they aimed to learn how to teach mathematics in language sensitive ways. They attended special seminars given by experts in the field of secondary language acquisition, which also included extensive coaching.

Most aspects of the evaluation of the project's impact concern the changes in the levels of the schoolchildren regarding their proficiency in German language, mathematical skills, and their ability to solve word problems. Proficiency in German language and mathematical skills were measured using standardised tests;

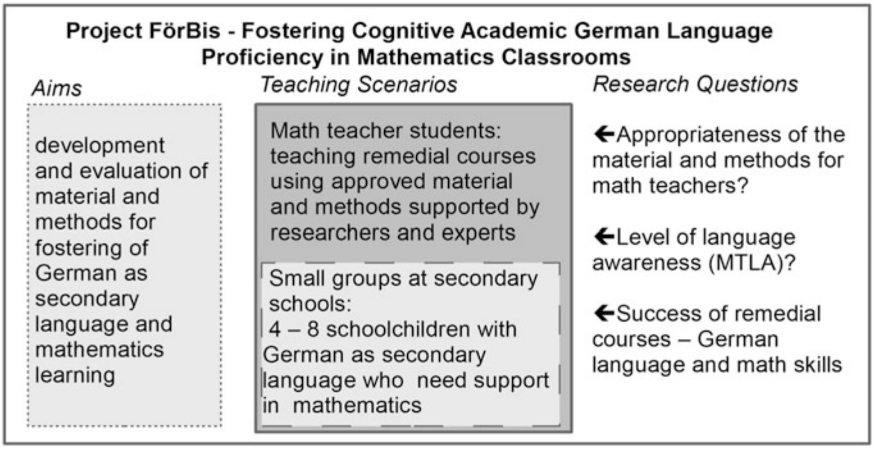

Fig. 1 Overview of Ph.D. project FörBis

to document the ability to solve word problems an informal test using the 'thinking aloud' method during the solution process and a qualitative analysis of this were used.

Concerning the level of the future mathematics teachers, the focus of the Ph.D. study is on the development of their language awareness regarding mathematics teaching. To document the changes in their knowledge, attitude, and awareness concerning the use of language in teaching mathematics, we asked the teacher students before they started their teaching year to create mathematical podcasts and to do a revision of their 'old' podcasts after nearly a whole year of teaching experience. In each of these phases, semi-structured interviews were conducted with the teacher students to supplement some information about their reasons and their intended use of the podcasts.

2 Theoretical Backgound

2.1 Language and Mathematics Learning

The role of language and mathematics learning in both aspects—'communication or meaning making' and in the context of learning mathematics in classrooms—is well known (Boulet, 2007; Cuevas, 1984; Meiers, 2010; NCTM, 2000; Schleppegrell, 2010). Recently in Germany many research projects have arisen in the context of schoolchildren with German as a secondary language.[1] Research questions generally focus on how to foster the learning of mathematics in special remedial courses or in regular classrooms. All of this is of great importance on the individual level as well as for changing society. In Germany mainly due to the arrival of exceptionally many refugees in 2015 and the implementation of the UN convention on inclusion, nearly all schools face the task of including a significant number of German as secondary language speakers in 'normal' classrooms of all subjects.

In Germany teachers in secondary schools major in two or three school subjects (i.e., mathematics, biology, and geography), but few teacher students combine mathematics with a language (mostly German, English, or French) for their majors. So usually mathematics teachers are not aware of the concepts of language acquisition like, for example, the difference between what Cummins (1979) describes as CALP (Cognitive Academic Language Proficiency) and BICS (Basic Interpersonal Communicative Skills) and the impact of these proficiencies on solving word problems in mathematics classrooms (Dyrvold, 2016). Further, many German mathematics teachers do not even see fostering secondary language acquisition as their responsibility (Becker-Mrotzek, Hentschel, Hippmann, & Linnemann, 2012). Also very often, mathematics teacher students and teachers

[1]See for example the research done at the Mercator Institute, at http://www.mercator-institut-sprachfoerderung.de/institut/english-information/, retrieved 03/11/2017.

alike are not aware of the obstacles they put in their students' paths by their own inattention to mathematics related language.

It may not be possible for all mathematics teachers to become experts in language teaching, but at least some support for language sensitive mathematics teaching is needed. In Germany there are now more and more projects (e.g., TMELL,[2] ProDaZ,[3] SPRACHE-AN-FACH-AN-SPRACHE[4]) and handbooks for teaching mathematics to secondary language learners (e.g., Celedon-Pattichis & Ramirez, 2012). It will take some time for these initiatives and the associated research to reach a broad number of schools in Germany.

2.2 Teacher Language Awareness

Using language—even the mother tongue—in mathematics teaching is much more than just knowing the (specific) vocabulary and the grammar. One major goal in mathematics teaching is to support students to develop appropriate mental models for mathematical concepts (Prediger, 2008; vom Hofe, Kleine, Blum, & Pekrun, 2005). Mental models ('Grundvorstellungen' in German) allow a person not only to visualise a mathematical concept but also to operate on it mentally. For example, if you have the mental model of a fraction—maybe, six eighths of a circle—then you can cancel this fraction mentally by combining pairs of the eighth-pieces to make one fourth-pieces. This would be one mental model for the cancelling of fractions. However, to communicate these models language in all its aspects is fundamental to understand mathematics.

Thus we want teachers to be aware of the impact of language on the teaching and learning of mathematics—their own language, their students' language, and the language in their schoolbooks. Andrews (2008) defined 'teacher language awareness (TLA) as a label applied to research and teacher development activity that focuses on the interface between what teachers know, or need to know, about language and their pedagogical practice' (Andrews, 2008, p. 287). This rather open definition, although originating in research on L2-teaching, can be easily adapted to mathematics teaching:

> Mathematics teacher language awareness (MTLA) encompasses what teachers know, or need to know, about language and the related pedagogical practice in teaching mathematics.

We did not choose a more precise definition, because we wanted this openness in our project.[5]

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[2]https://www.umb.edu/cosmic/projects/stem_ell/math_ell, retrieved 9/08/2017.

[3]Beese and Gürsoy (2011).

[4]http://www.leuphana.de/institute/ikmv/forschung-und-projekte/ag-fach-und-sprache.html, retrieved 9/08/2017.

[5]The Ph.D. project took place in a very complex setting as an explorative field study with teacher students and schoolchildren in a design based research approach. For example, each year the

2.3 Podcasts as Research Instruments

Podcasts—video- as well as audio-podcasts—are used more and more in higher education for different purposes (Fernandez, Simo & Sallan, 2009; Hew, 2009; Kay & Kletskin, 2012) and also for English language learners in mathematics classrooms (Fissore, 2011) or for German language learners in primary mathematics classrooms (Schreiber & Klose, 2014).

Audio podcasts are digital recordings, which can be downloaded to computers or other devices. Audio mathematics podcasts requires both the creator and audience to rely solely on spoken communication for explaining mathematical semantic content. Podcasts are not spontaneous but in our educational context are carefully planned and revised several times before they are considered 'finished'. Further, in our educational context, the creators document in a series of scripts or screenplays the process of creating a podcast.

3 Methods

3.1 Settings and Research Questions

In the FörBis project mathematics teacher students took part voluntarily and unlike in normal seminars, which last usually about 14 weeks, they had to sign on for one whole schoolyear. The mathematics teacher students worked in local secondary schools for 90 min each week with groups of 3–5 schoolchildren. The schools decided on the children in the fostering groups—which grade the children were from, the time of the remedial lessons, whether it was compulsory for the schoolchildren or not, etcetera.

So, the situation varied greatly for the mathematics teacher students. For example, some taught a compulsory group of five sixth graders, who attended frequently, others taught tree ninth graders in their final year who attended erratically.

All the mathematics teacher students were paid a small fee by the city of Ludwigsburg for their time at the schools and they all attended an accompanying seminar on language sensitive mathematics teaching for one semester and then came to office hours regularly for supervision. Also, teaching of the remedial courses is acknowledged as one of the three compulsory school internships ('professionalization internship') that teacher students at the Ludwigsburg University of Education have to serve.

number of participating teacher students and schoolchildren as well as their age groups varied. To develop a statistically firm standardized test for MTLA could not take place in this setting and would be another very important Ph.D. project.

The mathematics teacher students who choose to take part in the project were very few—over the three project years there were altogether 14 students. Obviously, they were not average students, but they were interested very much in the aspect of language and mathematics. (Otherwise, they would have taken the kind of seminars and internship with much less work.)

Because of the small number of participants, the very different settings in the schools and the bias in the mathematics teacher students an experimental design with treatment and control groups was not possible. Still we wanted to document and even measure the assumed change in their language awareness and especially their mathematics teacher language awareness (MTLA). Language related challenges or learning opportunities occur in mathematics teaching in very different ways from word problems with unclear meanings, mathematical definitions, to formal symbols, to wrong grammar or pronunciation. A good teacher with years of experience in language sensitive mathematics teaching should be able to support children adequately in any given situation.

But, how could we document or even measure this kind of mathematics teacher language awareness? First we tried to use mindmaps, which the teacher students created in the beginning of the seminar on language sensitive mathematics teaching. These mind maps focused on the role of language in mathematics teaching. The idea was to see whether there would be structural differences in the mindmaps on the same topic after spending some time on discussing and experiencing the language sensitive mathematics teaching. This idea proved to be not very informative because the students used more or less the same mindmap. Rather, the most interesting statements were given orally while they explained their mindmaps. So, we looked for an artefact that would focus on the oral language and allow us to document differences between the pre- and the postconceptions. These artefacts are the audio podcasts combined with semi-structured interviews, which form the basis of this study. The focus on audio, as opposed to video, podcasts requires the creators of the podcasts to rely solely on oral language. This also covers the aspect of teacher language in classrooms where everyday and academic language are often both present.

Since we did not have much experience with podcasts used for this aim, we did a pilot study to help us learn how to best handle the podcasts and to identify possible questions for the semi-structured interviews.

3.2 Pilot Study

Research question for the pilot study were:

- Is the effort of producing podcasts for teacher students adequate concerning the effects on the teacher students in terms of technical, time, and intellectual demands?

- Can the podcasts be analysed well for mathematics teacher language awareness (MTLA)? What time and effort does the analysis take?
- Are podcast a suitable artefact to document and measure MTLA?
- Are the central questions for the semi-structured interview regarding MTLA suitable for the target group and the research questions?

During the 2015 summer semester, both authors taught different seminars with different teacher students. One seminar on ‘Mathematics and (Secondary) German language’ addressed advanced teacher students who majored in mathematics (N = 5) and the other one was a freshman seminar on mathematics education generally (N = 41).

At the end of the seminars, we asked both groups of students to work in small groups of up to five persons to create audio podcasts on either ‘division’ or ‘multiplication’. They were loaned recording devices or used their own smartphones.

Audio podcasts in educational research are not just spontaneous recordings, but their creation follows a specific structure (similar to one described by Schreiber and Klose, 2014):

1. Spontaneous recording
2. Script 1
3. Recording following script 1
4. Time for some reading on the mathematical/educational topics
5. Revised script 2
6. Revised recording following script 2 → final version of the audio podcast

The students had about 90 min to work through this program.

By the end, we had nine audio podcasts, five on division and four on multiplication. The group of advanced teacher students choose the division topic. This was also the group who was interviewed using some central questions derived from theoretical literature (discussed in the next section).

The results of this pilot matched our expectations:

1. The creation of the audio podcasts was doable and motivating for all of the teacher students. Even the freshmen immediately started to reflect on language related challenges schoolchildren could face in mathematics.
2. We could distinguish clearly between the podcasts of the freshmen and the advanced teacher students regarding mathematics teacher language awareness. Table 1 shows the main differences between the two groups’ podcasts.

The advanced teacher students noticed already in the first review cycle of their scripts, that podcasts make it very difficult for schoolchildren to understand the mathematical topic. Therefore, they created some additional material on paper and did not depend on the oral level alone. Also, they tried out the podcasts with their private coaching students[6] and discussed the results in the next meeting. It was

[6]In Germany, a lot of teacher students give private coaching lessons to schoolchildren to earn some money.

Table 1 Differences between the podcasts of the freshmen and the advanced teacher students

	Freshmen	Advanced teacher students
Velocity/ enunciation	Fast/slurred	Slower/distinct
Wording	Typically CALP, specific mathematically	Using everyday expressions as much as possible
Grammatical structures	Complex, verbal parentheses[a]	Short sentences, few sub clauses
Mathematical description	Just description of the algorithms	Use of examples from realistic contexts representations for the mathematical concept of division Deliberate distinction between 'equal sharing' and 'equal grouping'
Focus	How to explain mathematical content purely orally—'without being able to show something'	How schoolchildren can understand mathematics

[a]Verbal parentheses are a typical German linguistic structure, where a verb is split into two parts in different places in a sentence. This causes German language learners many problems. i.e. to colour … anmalen: Der Junge **malt** schon den ganzen Tag Autos **an** (The boy is colouring cars for the whole day.)

notable that the advanced students followed the idea of helping the schoolchildren to reach the CALP-level while revising their first script (without any hints by the authors).

After piloting the creation and analysis of the podcasts and the first structured interview with the advanced students, we were confident, that these methods could be used to document and capture an expected increase in mathematics teacher language awareness (MTLA).

3.3 Methods Used in the Main Study

During the 2015/16 school year five teacher students majoring in mathematics participated in the FörBis-project. This means they agreed to teach remedial classes in a secondary school in Ludwigsburg for the whole school year and to attend a special seminar on teaching mathematics in language sensitive ways. They each taught small groups of two to six children in one age group. The remedial classes took place once a week during school times and took about 90 min each. The teacher students received appropriate material for fostering mathematics learning in a language sensitive ways and supervision in our special seminar on a weekly base.

Before they started in the schools and after a brief theoretical input about the appropriate use of German language in mathematics teaching for secondary

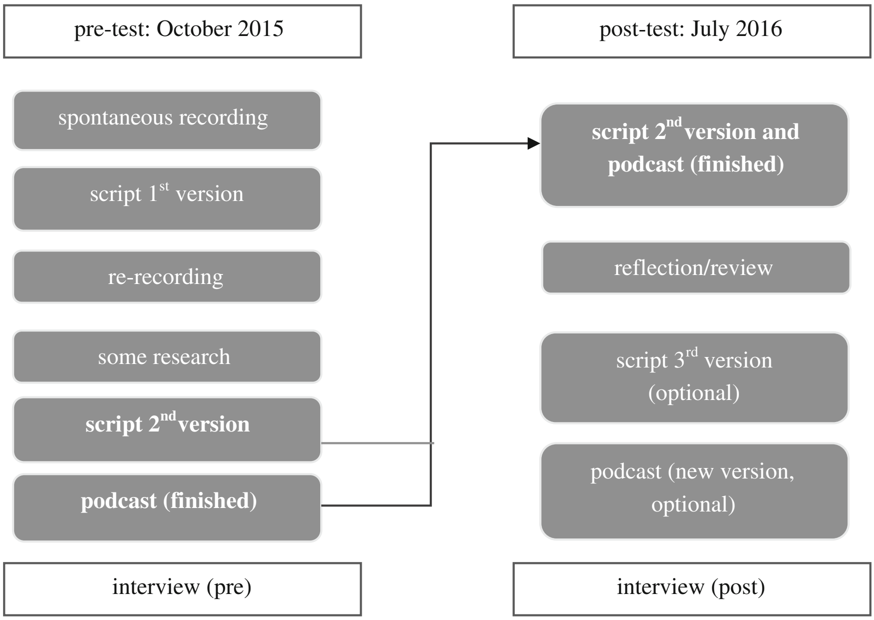

Fig. 2 The process of creating podcasts and interviews

language learners, these teacher students were split in two groups (N = 3 and N = 2) and asked to create a podcast on a given topic related to the mathematical content they would need for their remedial courses. These topics were 'fractions' (N = 3) and 'division' (N = 2). The process of creating the podcasts follows the description of a similar podcasts creation process by Schreiber and Klose (2014) who used podcasts created by mathematics teacher students for primary schools.

The main steps in the process were the alternations between oral recordings and written scripts as well as several revisions of the first spontaneous recordings. The reason behind this rather elaborate process is that these podcasts are intended to simulate a mathematics classroom in a small way focussing on the oral language. After the teacher students finished their podcasts each group was interviewed following the guidelines derived from similar studies and tested in the pilot study.

At the end of the school year, the teacher students were asked to reflect on their 'old' podcasts, and if necessary revise them and create new scripts and new versions of the podcasts. Afterwards they were interviewed again. Figure 2 shows the process of creating the different versions of podcasts and the interviews.

3.3.1 Semi-structured Interviews

After the participating teacher students finished the podcasts (pre-test) or they finished the reflection/revision of the 'old' podcasts (post-test) the two groups were

Table 2 Interview guidelines

Guidelines for semi-structured interviews (English translation)
1. Structure and rationale behind the podcast creation process
What approach did you take to create the podcast? Why did you decide on this approach?
Did you change anything in the podcast creation process regarding the approach? (only post-test)
2. Meaningful elements of podcasts
What did you pay attention to primarily while creating the podcast?
What advice would you give other teacher students creating podcasts?
What should they pay attention to absolutely?
What did you pay attention to revising your podcast? (only post-test)
3. Evaluation criteria for strong or very strong podcasts
What should a podcast be like for you to consider it 'strong' or 'very strong'?
4. Classroom use
Would you use the podcast in a classroom? Why?
Under what conditions would you use the podcast in a classroom?
What would a possible use of podcasts in your classroom look like?
5. Evaluation of the process (helpful steps)
Which steps were particularly helpful creating the podcast?
Which steps were particularly helpful revising the podcast? (only post-test)

interviewed separately. These interviews were recorded and transcribed word-for-word using a simple transcription method. Table 2 shows the English translation of the interview guideline (for the original German interview guideline see Appendix). The interviews were conducted in German, the mother tongue of this group of teacher students and the teaching language at the university. For both groups all steps were documented and all artefacts (scripts and recordings) were made available to the researchers.

4 Data Analysis

4.1 Podcasts

Versions 2 and 3 of the podcast scripts from both groups were analysed according to a category system focusing on word and sentence levels, verbal expressions, and formulation variations (Prediger, 2014). The intended comparison of the two podcast versions was abandoned, because one group ('fractions') did not change anything at all. And the other group ('division') only changed very little—mainly correcting a sentence with a mistake.

4.2 Semi-structured Interviews

The recorded interviews were transcribed word-for-word by the second author and two student research assistants, who had written their master theses on the topic 'Language Sensitive Mathematics Teaching'. For the transcription, the software 'f4' (https://www.audiotranskription.de/english) and simple transcription methods were used.

The transcripts were analysed using Mayring's (2014) qualitative content analysis. The deductive categories were taken from the interview guideline. In addition, an analysis of inductive categories based on the data was conducted (and it still is because the Ph.D. project is still ongoing). For the analysis, the software MaxQDA (http://www.maxqda.com) was used. The second author did the qualitative content analysis and identified the instances of the deductive categories system as well as identifying inductive categories.

5 Preliminary Results

Currently, the qualitative content analysis of the semi-structured interviews is not finished, but we can give some exemplary inside views on 'classroom use.' For this, the questions in the interview guidelines were:

- Would you use the podcast in a classroom? Why?
- Under what conditions would you use the podcast in a classroom?
- What would a possible use of podcasts in your classroom look like?

Because the interviews are in German, we do not give the anchor examples and the students' answers verbatim. Table 3 gives the summary of the pre- and post-test statements in the interviews with the 'fraction'-group. In the right column, there is a first interpretation with a focus on the shift of interest between the pre- and the post-test.

In pre-test interviews, the teacher students focused mainly on the idea that teachers supply the class with readymade podcasts on different mathematical topics. The teachers are responsible for the correctness of the podcasts mathematically and language wise. Whoever needs some explanation on specific topics choses the podcast and listens to it. Schoolchildren creating their own podcasts are considered motivated due to the more active role of the creator. On the other hand, the teacher students doubt whether mathematical topics interest schoolchildren at all.

In the post-test interviews, the teacher students describe the oral presentation of mathematical content as a matter of course from the viewpoint of the schoolchildren. They focus on challenges the schoolchildren will face regarding the language aspects in mathematics learning. Further, they mention possible support ideas like 'explain videos'. These are videos created by teachers or even pupils beforehand, which explain certain mathematical concepts or processes—for example, the

Table 3 Pre- and post-test comparisons of classroom use of podcasts

Classroom use of podcasts		
Pre- and post-test comparison in the 'fraction'-group (N = 3)		
Statements Pre-test (October 2015)	Statements Post-test (July 2016)	First interpretation Shift of interest
Teachers as active agents		
Time and effort creating podcasts versus little time resources Technical equipment necessary Motivating instrument for schoolchildren to listen to the podcast Schoolchildren as recipients can use the podcasts individually to work on their deficits in different mathematical contents ('teacher replacement') Mathematical correctness is the responsibility of the teacher	Teachers should be able to record purely oral podcasts easily, because 'explaining mathematics' is their every day job Explanations have to be deliberately prepared according to the mathematical content and the phrasing Explanations occur at a certain [academic] level, but they should not be filled by technical terms only	**Pre-test** Focus on teacher activities to ascertain correct mathematical content Schoolchildren as recipients of pre-produced podcasts **Post-test** Teacher activities described are focusing on the classroom level Lesson planning and learner support are the central issues for realising the cognitive demands of mathematical contents
Schoolchildren as active agents		
Motivation as factor for using podcasts in mathematics classrooms Podcasts used as part of 'mathematical library' in the classroom Focus on mathematical correctness Active participation possible instead of passive reception of explanations if schoolchildren would create their own podcasts Doubtful whether mathematical topics could be motivating for the schoolchildren at all to create own podcast	General phrasing of mathematical facts need to be understood by learners 'Explain-videos' are preferred to purely oral podcasts because they allow visualisations supporting oral explanations Focus on students' phrasings Teachers responsible for support concerning language issues, i.e., giving beginnings of sentences or sentence patterns	**Pre-test** Learners' use of podcasts mainly justified by motivation to do something 'different' **Post-test** Focus on learners' activities 'Phrasing' seen as mathematical activity not only for motivational reasons but fundamental for learning mathematics Thoughts on the high level of cognitive requirement of oral explanations in mathematics 'Explain-videos' instead of podcasts to use visualisations

multiplication algorithm.[7] Also, the need to get schoolchildren to express mathematical relations as clearly as possible is regarded as an important positive side-effect of schoolchildren creating mathematics podcasts.

[7]The teacher students prefer videos to audio podcasts for this 'explain' aspect, because with videos they can 'show' something—for example, how long multiplication is done.

6 Discussion

The preliminary results represent one perspective on the transcripts, which will be complemented by full analysis. The qualitative content analysis of the semi-structured interviews will be a major chapter in the Ph.D. thesis of the second author. We return to the research questions that apply to the whole project to see how they can be answered with this perspective on the data:

- Is there a change/increase of mathematics teacher language awareness (MTLA) after one year of teaching remedial classes and attending a seminar?
- Are podcasts effective for documenting or even measuring a change in the MTLA?

Both questions can be answered with a careful ‘yes, but …’.

The first results of the analysis of only one category of the semi-structured interviews clearly indicate that there is a shift of interest in the group of mathematics teacher students toward an increased MTLA. They noticed the importance of what the schoolchildren say about mathematics, they see the cognitive challenge connected to phrasing mathematical facts for German language learners, and they recognize the importance of well-planned mathematics lessons—even on language aspects.

Are the podcasts a suitable tool for documentation of changes in MTLA? As seen in the pilot study the podcasts themselves allow us to differentiate between people who are new to the topic of language sensitive mathematics teaching and people more experienced with the topic. But, the podcasts do not allow us to document directly the change in MTLA of one person or a group over time. One reason may be that even people with a little experience can create a very good podcast. As always, what matters mostly is how one uses the podcasts or other material in the classroom settings.

But the *process* of creating the podcasts works immediately as a very good prompt for reflection on and talk about the use of language in mathematics classrooms. Based on this immediate effect, the semi-structured interviews about the creation and use of podcasts in mathematics classrooms are very good instruments to document changes in MTLA.

Future research may develop a more standardized test on the MTLA, perhaps using the creation of podcasts in combination with a semi-structured interview or an open response questionnaire. Based on this ‘MTLA-test’ different scenarios for teacher students or for working teachers could be evaluated for their effects on the increase of MTLA.

Acknowledgements The Mercator Institute, Cologne, Germany funded the research described in this chapter.

Appendix: German Version of the Interview Guidelines

Leitfragen im Pre-Test und im Post-Test zu den Podcasts

1. Aufbau und Begründung der Vorgehensweise bei der Erstellung des Produktes (Podcast)
Für welche Herangehensweise haben Sie sich entschieden? Wieso haben Sie sich für diese Herangehensweise entschieden? Haben Sie an dem Aufbau Ihres Podcasts in Bezug auf die Herangehensweise etwas verändert? (Nur im Posttest)
2. Bedeutungselemente von Podcasts
Auf was haben Sie bei der Erstellung besonders geachtet? Was würden Sie anderen Lehreranwärtern für die Erstellung von Podcasts raten? Was sollten sie unbedingt beachten? Auf was haben Sie bei der Überarbeitung besonders geachtet? (Nur im Posttest)
3. Bewertungskriterien guter oder sehr guter Podcasts
Wie müsste ein Podcast sein, damit Sie es als „gut“ oder „sehr gut“ bewerten?
4. Unterrichtseinsatz
Würden Sie Podcasts im Unterricht einsetzen? Warum? Unter welchen Bedingungen würden Sie Podcasts im Unterricht einsetzen? Wie könnte ein möglicher Einsatz von Podcasts bei Ihnen im Unterricht aussehen?
5. Bewertung des Prozesses (Hilfreiche Schritte)
Welche Schritte waren für Sie besonders hilfreich bei der Erstellung? Welche Schritte waren für Sie besonders hilfreich bei der Überarbeitung? (Nur im Posttest)

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Exploring How a Grade 7 Teacher Promotes Mathematical Reasoning in a Multilingual Mathematics Class of English Second Language Learners

Faith Lindiwe Tshabalala

Abstract This qualitative study was conducted in a school in an informal settlement West of Johannesburg. The study explored how a grade 7 teacher promoted mathematical reasoning in a multilingual mathematics class of English second language learners. The focus of the research was on how a Grade 7 mathematics teacher interacted with the learners to encourage mathematical reasoning. The study was informed by a theory of learning which emphasises the importance of social interaction in the classroom where the teacher encourages learners to interact with each other to explain their thinking and to justify their answers. The analysis shows that the teacher focused on developing the learners' procedural fluency and that this focus on procedural fluency was accompanied by the dominant use of English by the learners.

1 Introduction

Mathematical reasoning is one of the important aspects of the new curriculum in South Africa (Department of Education, 2002). Learners are expected to actively question, examine, conjecture, and justify their solutions and present arguments. Educators are thus expected to promote these practices when teaching mathematics. Teachers have to ensure that the learners are exposed to mathematical practices that promote mathematical reasoning. Furthermore, the Revised National Curriculum for Mathematics General Education and Training indicates that the mathematics programme should provide opportunities for learners to develop and employ their reasoning skills and be able to evaluate the arguments of others (Department of Education, 2002).

The purpose of this study was to investigate whether and how a Grade 7 teacher in a multilingual class encouraged and facilitated mathematical reasoning in a Grade 7 class of English second language learners. The study was guided by the

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© Springer International Publishing AG, part of Springer Nature 2018
J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_18

following questions: (1) How does a Grade 7 mathematics teacher in a multilingual class encourage mathematical reasoning during teaching? (2) How did teacher interactions with the learners enable or constrain mathematical reasoning?

The study focuses on mathematical proficiency and especially mathematical reasoning. Mathematical reasoning is not a content area like addition; it is an activity that is embedded in the practice of mathematics and therefore not easy to teach. Kilpatrick, Swafford, and Findell (2001) propose that mathematical proficiency includes mathematical reasoning. Mathematical proficiency is evident when learners show conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition (Kilpatrick et al., 2001). Ball and Bass (2003) maintain that while promoting mathematical reasoning there should be interaction in the classroom. That interaction involves language, which could enable or constrain mathematical reasoning. In other words, language cannot be avoided in a mathematics classroom in which mathematical reasoning is encouraged.

2 Theoretical Framework and Literature Review

The study is broadly informed by a social constructivist theory of learning (Taylor & Campbell-Williams, 1993 cited in Jaworski, 1994; Von Glasersfeld, 1987; Vygotsky, 1978), which recognises the importance of a knowledgeable other in the construction of knowledge. To understand how the teacher promoted mathematical reasoning in the learners, the five strands of mathematical proficiency identified by Kilpatrick et al. (2001) were used to analyse the interaction during the lesson. Kilpatrick et al. (2001) argue that when learners are mathematically proficient it means their mathematical reasoning has developed.

The five strands of mathematical proficiency are: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, as well as productive disposition. While the teacher can promote mathematical reasoning using the above-mentioned strands, an issue of social interaction arises especially when learners have to explain and justify their answers. This is supported by Taylor and Campbell-Williams (1993), Von Glasersfeld (1987); Vygotsky (1978); cited in Jaworski (1994).

According to Ball and Bass (2003) mathematical reasoning enables mathematical understanding. This implies that mathematical understanding is founded on reasoning and mathematical reasoning is fundamental to using mathematics (Ball & Bass, 2003). They further emphasize that mathematical reasoning enables learners to tackle problem solving even if they have forgotten an algorithm. In other words, this suggests that assessing the validity of learners' ideas presents a number of challenges for teachers. For teachers to know how and when to probe particular learners' ideas, they need to be able to identify where there is potential for productive mathematical reasoning (Brodie, 2005).

Fraser, Murray, Hayward, and Erwin (2004) maintain that learners should be given tasks that encourage them to work co-operatively, discuss, argue and reflect on the methods of others. Talking and hearing others involves language; through talking the learners make their mathematical knowledge public and usable by the collective (Ball & Bass, 2003). This is why Ball and Bass (2003) argue that language is the foundation of mathematical reasoning.

Moschkovich argues that mathematics classrooms are shifting from a focus on primarily silent and individual activities as "students are now expected to communicate mathematically, both orally and in writing, and participate in mathematical practices, such as explaining solution processes, describing conjectures, proving conclusions, and presenting arguments" (2002, p. 190). To ensure fruitful interaction in a mathematics classroom; teachers have to ask learners to justify their answers (Ball & Bass, 2003). Stein, Smith, Henningsen, and Silver (2000) argue that if the teacher wants students to learn how to justify or explain their solution processes, he/she should select tasks that are deep and rich enough to afford such opportunities. They contend that tasks should have the potential to engage learners in complex forms of thinking and reasoning. Stein et al. (2000) suggest 'doing mathematics tasks' as tasks that can promote mathematical reasoning because they require complex and non-algorithmic thinking.

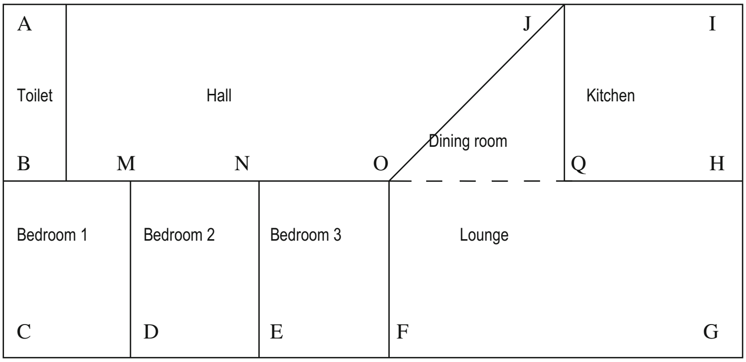
3 Methodology

This qualitative case study focused on one multilingual Grade 7 classroom in a township school west of Johannesburg. The language of learning and teaching in the school is English; however none of the learners or the teachers in the school had English as their main, home or first language. Data in this study was collected through lesson observation and interviews. The teacher was interviewed twice during the study. The lesson transcript was coded using the five strands of mathematical proficiency identified by Kilpatrick et al. (2001). Utterances in the transcripts were counted based on the categories above.

3.1 A Brief Description of the Lesson Observed

The lesson was based on a task with a real life context of money focused on the concepts of shape and space. The task had five questions. The teacher gave learners an opportunity to work on the task in groups and after each question he interacted with them in a whole class discussion about the ways they had solved the problem. He arranged learners in groups of four. The teacher did not give the entire task to the learners at once, but instead gave them one question at a time.

He read and explained each question to the learners, in some instances he switched to Setswana to ensure that all the learners understood. He read the questions,



AB = 3m FG = 8.5m
BC = 2.5m HI = 3m
CD = 4m OJ = 5m
DE = 3m QH = 4.5m
EF = 3m KP = 7m
PJ = 4m

1. Find the area of the house.
2. Find the area of the hall.
3. If a square metre box of tiles has five tiles, how many boxes will you use in the kitchen?
4. How much will you pay the person who will put the tiles in the hall if he charges R30 per square metre?
5. What will be the size of the carpet to be used in the bedrooms?

Fig. 1 Task

explained briefly what was required, gave hints through probing questions, and then gave students an opportunity to work in groups. At the end of each question he requested one member from each group to explain how that particular group arrived at the solution. Below is a copy of the task as it was given to the learners (Fig. 1).

3.2 *The Teacher's Role in Promoting Mathematical Reasoning*

To analyse data I first counted the utterances of teacher-learner interaction in each of the categories. This was done before analysing the lesson observed. Below is the table that shows the categories used to analyse the lesson, teacher-learner interaction, language use, as well as the number of utterances within each category in the transcript. The kind of tasks the teacher uses to promote mathematical reasoning according to Stein et al.'s (2000) analyses of mathematics instructional tasks are described in Fig. 2.

Levels of Demands (Smith and Stein 1998).
Lower-level demands (Memorization): Involve either reproducing previously learned facts, rules, formulas, or definitions or committing facts, rules, formulas or definitions to memory. Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure. Are not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated. Have no connection to the concepts or meaning that underlie the facts, rules, formulas, or definitions being learned or reproduced.
Lower-level demands (Procedures without Connections): Are algorithmic. Use of the procedure either is specifically called for or is evident from prior instruction, experience, or placement of the task. Require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it. Have no connection to the concepts or meaning that underlie the procedure being used. Are focused on producing correct answers instead of on developing mathematical understanding. Require no explanations or explanations that focus solely on describing the procedure that was used.
Higher-level demands (Procedures with Connections): Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts. Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations helps develop meaning. Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.
Higher-level demands (Doing Mathematics): Require complex and non-algorithmic thinking—a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example. Require students to explore and understand the nature of mathematical concepts, processes, or relationships. Demand self-monitoring or self-regulation of one's own cognitive processes. Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task. Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions. Require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution process required.
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Fig. 2 Characteristics of tasks at different levels of cognitive demand

3.3 How Did the Teacher Develop Mathematical Proficiency?

In this section I focus on how the teacher worked with the learners to develop the five strands of mathematical proficiency.

3.3.1 Conceptual Understanding

The number of utterances identified in my analysis shows that promotion of conceptual understanding was minimal in this lesson. Conceptual understanding in learners was seen when the teacher demanded that they explain how they got 5.5 m. Asking how the learners got 5.5 m the teacher was assessing the learners' understanding of the concept of measurement. The learners managed to relate the 5.5 m to AB and BC. In line 78 below the teacher was promoting sense making and deeper levels of understanding by constantly asking the learners to give the reason why they had to add 3 and 2.5 m together.

The teacher's encouragement of the learners to justify and explain their answers through questioning helped them to explain how they got 5.5 m and why they had to add 3 and 2.5 m. For example, Jeremiah (line 79) linked AB and BC with the measurement of the breadth and seemed to understand the meaning of breadth and how it is represented in a diagram.

74	T:	How did we get that 5.5 m. Who can explain? I think we've explained thoroughly now, we want somebody who can stand up, explain it to us in full, how, okay, let's hear. Can we listen to him, can we listen to him, yes.
75	L:	AB is 3 m, BC, 2.5 m
76	T:	Then? What do we do with the two measurements?
77	L:	You say 3 + 2.5 is 5.5 m.
78	T:	What is the reason? Why do we do it? Remember in maths all the time ask the reason 'why do we do that'. Why 5.5 m, it is correct, the answer is correct but now why, what is it anyone except Percy, Percy, okay Jeremiah, yes.
79	Jeremiah:	because we want to get the breadth.

Kilpatrick et al. (2001) argues that teachers cannot ask conceptual questions if they do not teach for conceptual understanding. In other words, there has to be a connection between the kinds of questions teachers ask and the way they teach. The teacher made it possible for the learners to display the capacity to engage conceptually about the concepts that they were learning through the questions he asked.

3.3.2 Procedural Fluency

Kilpatrick et al. (2001) describe the procedural fluency strand as the knowledge of procedures, how and when to use them appropriately and the skill in performing them flexibly, accurately and efficiently. Procedural fluency is thus not limited to the ability to use procedures; it also includes knowing when and how to use them.

The teacher in the study supported procedural fluency by ensuring that they knew when and how to use procedures. The number of utterances identified in my analysis and in the transcript show that procedural fluency dominated the entire lesson. In the extract below the teacher asked the learners to explain what it was they were multiplying and how they had multiplied. The learners were able to put all the points that make the length and the breadth together, i.e. 18.5 m for length and 5.5 m for breadth. In lines 94 and 98 the teacher required that the learners explain the procedures they followed when multiplying the length and the breadth. He wanted them to explain how they arrived at the result of 101.75.

92 T: Ya the length and the breadth akere [okay] now what are the measurements of that length and the breadth
93 L: The length is equal to 18.5 m and the breadth is equal to 5.5 m
94 T: Okay, now explain to me how did you multiply?
95 L: We said $5 \times 5 = 25$ carry the 2, $5 \times 8 = 40 + 2 = 42$, $5 \times 1 = 5 + 4 = 9$
96 T: Okay
97 L: And then it give us 101.75 m
98 T: 101.75 m, usually numbers after the comma we call them one by one ekere [Okay]? Eh, let's see the last group, in this case okay who's explaining here now the first thing what is it we were doing?
99 L: 18.5 m $\times$ 5.5 m
100 T: Okay?
101 L: After that we said that $5 \times 5 = 25$ carry the 2 and then 8×5 is 40 plus 2 is 42, 1 times 5 is 5, 5 plus 4 is 9…

In the above extract, the teacher asked the learners to describe how they had performed basic computations. The calculations required accurate mental arithmetic and a flexible way of dealing with procedures. Line 94 shows the teacher was requiring the learners to explain how they multiplied, in other words he was asking for a description of the procedure they used in order to get to the answer. In line 95 the learner described the procedure, doing mental calculations at the same time. This is evidence that the teacher supported procedural fluency during this lesson.

3.3.3 Strategic Competence

Strategic competence refers to the ability to formulate, represent, and solve mathematical problems. Kilpatrick et al. (2001) argue that it is crucial for mathematical proficiency that learners should know a variety of solution strategies as well as which strategies might be useful for solving a specific problem.

As much as there is evidence that the teacher asked the learners to explain how they solved the problems (see line 230 and 232 below), the number of utterances identified in my analysis shows that the teacher gave little effort to promoting strategic competence. In line 230 the teacher asked the learners to come up with a strategy that they would use in order to find the size of the carpet for the three bedrooms by saying, "he wants to put the carpets, what will he do?" Instead of giving the learners the answer, he gave them clues so that they could produce their own strategies. This suggests that the teacher was trying to support the learners in developing their own strategies for solving mathematical problems. In line 232 he asked them how they were going to solve the problem, in other words he wanted them to explain the strategy they would use to get the size of the carpet. In line 230 and 232 he used phrases like 'what is it that we should do?', 'what will he do?', 'how are we going to determine the size of the carpet?' He also asked for an explanation of the strategy, saying 'okay Percy can you explain'.

230	T:	Okay that's good. Now let's do the last one. {reads the last question on the chalkboard} What will be the size of the carpet used in the bedrooms? Now obatla ho kenya carpet mo di bedrooms tsa haye akere [he wants to put them in his bedroom, okay], now what is it that we should do? Who can tell me? He wants to put the carpets in the bedrooms, otlo yetsang [what will he do]? What is it that we should do? The carpets! Now where are the bedrooms {pointing at the diagram} how many bedrooms do we have?
231	Learners:	Three
232	T:	And we want to put what? Can you tell us without writing, what is it that we should do? Retlotseba joang gore di carpet detshwanetse debe kakang [how are we going to determine the size of the carpet]? Otloyetsang [what will he do]? What is it that we should do, otlo tseba joang [how will he know], okay Percy, can you explain to us?
233	Percy:	We find area from CF and BO.
234	T:	Simple as that, now what is the area, what is the area? Remember guys, the area, what is the area?

3.3.4 Adaptive Reasoning

Adaptive reasoning is the capacity for logical thought, reflection, explanation and justification. This strand was the least evident in this lesson, as identified in my analysis and shown in the transcript below. Adaptive reasoning was most evident when the teacher was dealing with the question on money. In line 227 below the teacher required Gugu to explain how she got 810.

226 T: That row, we want somebody now, so far you have done well. Somebody new, Gugu can you try? Can you explain it to us? Can we…

227 Gugu: 30 × 27, then we said 7 × 0 is 0, 7 × 3 it's 21 then we write 21 then we have 20 is 0 under 1 and 2 × 3 is 6 then we add 210 + 60 then it gave us 810. {pause}. I said 0 × 27, 7 × 0 is 0, 7 × 3 is 21 then I have 210. Then I say 2 × 3 is 6 then I have 60 then I say 210 plus {pause, looking at 60 and the gap in units} I think I must put 0 here under the units because I cannot write a number like this, I must write units which is 0. {she then had 210 + 600 like this:}

$$\begin{array}{r} 30 \\ \times\ \underline{27} \\ 210 \\ +\ \underline{600} \\ \underline{810} \end{array}$$

224 T: Why did you write the 0? Explain to her. Jeremiah explain to her, why did you write that zero?

225 Jeremiah: If we say 210 plus 60 it won't give us 810

226 T: Quiet class, ja [yes] tell us where did you get the 600?

227 Jeremiah: mh…. Eh… 2 is in 27, 2 is tens then is 20, we say 20 × 30 is 600.

228 T: So what will be the total now?

229 Jeremiah: Total will be 810

During the interview, the teacher stated that he preferred tasks that provoke learners to think and the teacher mentioned the characteristics he would want to see in the task that promotes mathematical reasoning:

1 Researcher: What kinds of tasks or mathematical activities do you use to promote mathematical reasoning in your class?

2 Teacher: … Tasks should require or force them to think, they should talk to one another if necessary, discuss, and the task should drive them to use or recall other maths concepts to find solution.

Despite the fact the teacher mentioned that the task should force the learners to think, to talk to one another, discuss, to recall, to reason, to give explanation with confidence and be able to explain the procedures they have used, these were not evident in his lesson. The pre-interview showed that the teacher was aware that there should be a link between his understanding about the tasks that are relevant for promoting mathematical reasoning, the way he teaches and the kinds of questions he asked to assess adaptive reasoning in learners, in this lesson however he does not strongly challenge adaptive reasoning. This suggests that the teacher needed the skill of assessing the mathematical validity of learners' ideas or methods of solution (Schifter, 2001). Schifter (2001) points out that the teachers' attention needs to be continually drawn back to the mathematics in what children are saying and doing and so that he may be able to help them solve mathematical problems on their own.

3.3.5 Productive Disposition

According to Kilpatrick et al. (2001), learners have developed productive disposition if they have developed a tendency to see sense in mathematics, and perceive it as both useful and worthwhile. Since there was no data collected on the learners' beliefs, conclusions could not be drawn regarding this strand.

3.4 Probing the Learners to Talk About Mathematics

The teacher intervened consistently throughout the lesson by probing learners to talk. Stein et al. (1996, p. 457) argue that, "Students must be first provided with opportunities, encouragement and assistance to engage in thinking, reasoning and sense making in the mathematics classroom." The teacher set up classroom tasks and group work situations where learners engaged with the task and then gave the learners an opportunity to express their thinking to the whole class. He attempted to engage learners throughout the lesson and asked them to talk about their mathematical solutions. However, the content of the discussions focused on procedures not concepts.

He rarely switched to Setswana or isiZulu when explaining questions or scaffolding the question. In the extract below the teacher asked the learners to talk about the procedures they were using. He did not accept a result; instead he used the answer to pose another question. Through the help of the teacher, learners were able to explain why and how they had used a particular formula or algorithm.

80	T:	Now from there what do we have? Because we've got, from there we have our length which is 18.5 m times and what is our breadth?
81	L:	5.5 m
82	T:	Then from there what do we do? Once we have substituted now akere [not so], we say L actually stands for what?
83	Learners:	18.5 m
84	T:	So B stands for what?
85	L:	5.5 m
86	T:	Then what do we do from here? Ya
87	L:	We multiply the length by breadth.

In lines 80, 82, 84, 86 the teacher accepted the answer and then probed further. Each response led to the next step in a procedural description of the solution. Through this probing process, the teacher was able to engage the learners in describing their procedure. The teacher supported a focus on procedures while probing students to talk. When he probed the learners to talk in the above extract, he first hinted at a procedure and then asked students continue to describe the procedure. In doing so the teacher in fact reduced the cognitive demand of the task and used only procedural questions. It is clear from the analysis following the characteristics in Fig. 2 that how the teacher used language to talk about mathematics made the focus of the lesson predominantly procedural. The above extract illustrates how the teacher's probing limited the learner-talk as the learners were responding with one word or short sentences which focused on procedures. Such short procedural responses make it difficult for the teacher to assess the mathematical validity of the learners' ideas as there is little evidence of the sense-making in their mathematical thinking (Schifter, 2001).

4 Summary of the Findings

4.1 Did the Teacher Encourage Mathematical Reasoning? How Did the Way in Which the Teacher Interacted with the Learners Enable or Constrain Mathematical Reasoning?

The teacher selected a task that would enable the promotion of mathematical reasoning—with lower level demand questions and higher-level demand questions. The task had the potential to support the five strands of mathematical proficiency (Kilpatrick et al., 2001). The implementation however, focused mainly on procedural fluency more than the other strands. During this lesson, the teacher focused more on procedural fluency than the other five strands of mathematical proficiency. Although he constantly asked the learners to explain, his probing led to the descriptions of procedures, not to students' mathematical reasoning.

While there was evidence of descriptions of solutions, there was no evidence of conjecturing, justifying, or presenting mathematical arguments. Although he tried to ask learners to justify their answers, the focus on procedures took the emphasis away from the challenging aspects of the task (Stein et al., 1996). Although he did open up for conversational space for learners to make their ideas public, most of the questions were procedural.

A focus on procedural fluency can deprive learners of the opportunity to conjecture, present arguments, and justify their conclusions. Carpenter, Franke, and Levi (2003) support the importance of conjecturing by stating that they have "found that it is productive to ask children whether their conjectures are always true and how they know they are true" (p. 102).

5 Conclusions and Recommendations

Promoting mathematical reasoning is a challenge in teaching mathematics. This study illustrates how difficult it can be to focus a lesson on students' mathematical reasoning. Such lessons require attending to the mathematics in what the learners are saying, assessing the mathematical validity of the learners' ideas, listening for the sense in the learners' mathematical thinking even when something is missing, and identifying the conceptual issues the learners are working on (Schifter, 2001).

This study points to the importance of learner talk in promoting mathematical reasoning and how the teacher facilitates the process. It is not sufficient for a teacher to select an appropriate task. How a task is implemented is also critical. As Schifter (2001) argues, before teachers give a task to learners they should consider the following:

- How to attend to the mathematics in what the learners will be saying and doing (Schifter, 2001, p. 120f).
- How to assess the mathematical validity of learners' ideas (Schifter, 2001, p. 121f).
- How to listen to the sense in learners' mathematical thinking even when something is amiss (Schifter, 2001, p. 126f).
- How to identify conceptual issues the learners are working on (Schifter, 2001, p. 128f).

Schifter (2001) argues that though teachers may have strong mathematics background they still need to learn to attend to learners' mathematical thinking. They need to be able to listen to learners with sharpened curiosity and interest, know which questions to ask when learners respond, and know how to support the learning process (Schifter, 2001).

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The Meaning of ‘Number’ in Kaiabi Language: Indigenous Teachers’ Identity Discourses in a Multilingual Setting

Jackeline Rodrigues Mendes

Abstract This study is based on ethnographic research carried out in a multilingual context in the Xingu Indigenous Park in a setting of indigenous teachers’ education. The discussion focuses on identity and language issues, examining the process of developing a mathematics textbook written in indigenous language by Kaiabi teachers to be used in indigenous schools in the Park. This discussion explores the Kaiabi cultural meaning of ‘number’ in community practices and the practices arising from contact with non-indigenous society. The aim is to point out the relationship between knowledge, language, and identity, and the tension that is established, specifically in the indigenous context, when indigenous written language has assumed a symbolic character in the assertion of ethnic identities.

1 Introduction

This chapter is based on ethnographic research carried out in a multilingual context in the Xingu Indigenous Park and developed over five years in a setting of indigenous teachers’ education. The Xingu Indigenous Park is located in the central region of Brazil in the State of Mato Grosso and has an area of 6,532,158 acres. The Indian reservation is crossed by the Xingu River and is divided in two regions called Lower Xingu (in the North) and Upper Xingu (in the South). Fifteen ethnic groups currently live there: Kuikuro, Kalapalo, Matipu, Nahukuá, Mehinaku, Waurá, Aweti, Kamaiurá, Trumai, Ikpeng, Yawalapiti, Suya, **Kaiabi**, Yudjá and Kayapo. The discussion focuses on identity and language issues, examining the process of developing a mathematics textbook written in indigenous language by Kaiabi teachers to be used in indigenous schools in the Park.

I present here some results of my research, spawning a discussion about the meaning of ‘number’ for Kaiabi teachers. This discussion explores the Kaiabi cultural meaning of ‘number’ in community practices and the practices arising from

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J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2_19

contact with non-indigenous society. It highlights the symbolic and political aspects present in the numerical terminology expansion of indigenous language by the teachers, elders and community leaders to produce a Kaiabi mathematics textbook. In an etnomathematical perspective this chapter investigates the Kaiabi cultural meaning of "number" within the broader identity politics of indigenous schooling. The aim is to point out the relationship between knowledge, language, and identity, and the tension that is established, specifically in the indigenous context, when indigenous written language has assumed a symbolic character in the assertion of ethnic identities, with each group aiming to be recognized not only as 'indigenous' but as an Indian Guarani, Kaiabi, Kamaiurá, Tupari, Waiãpi, Xavante, and so on, out of about 200 ethnic groups existing in Brazil.

Historically, indigenous school education in Brazil was marked by two prevailing total immersion programs. In one, indigenous children were taken away from their environment and submitted to a monolingual education in Portuguese, according to an official school curriculum. In transition programs, bilingual educational practice was advocated, although the indigenous language was actually given a subordinate role, being a means of instruction in Elementary School until the necessary proficiency in Portuguese was attained, the presence of indigenous language in the curriculum was aimed at the acquisition of Portuguese. In opposition to such models, a third one has established itself in the search for an indigenous schooling aimed at linguistic preservation and cultural maintenance of indigenous peoples (Cavalcanti & Maher, 1993). The search for such a model led to a discussion about the need for a specific, differentiated, intercultural and fully bilingual indigenous school (Mendes, 2001, 2007).

This concern with cultural specificities and autonomy acquisition, together with the question of socio-cultural and linguistic diversity of indigenous groups in Brazil, drew attention to the necessity of having indigenous teachers work in the schools and led to the development of education programs to prepare these teachers. These programs began in the 80s through projects developed by non-governmental organizations. These projects were developed with the participation of consultants from universities, from different fields such as Anthropology, Linguistics and Education. In the mid 90s, other programs were constituted by the action of state education bureaus and universities. The attempt to create and establish bilingual, intercultural, and culturally specific indigenous schools resulted in a curricular discussion process (Ministério da Educação, 1998) and the increased development of bilingual materials by the indigenous teachers in the process of their teacher education.

This chapter focuses on an indigenous teacher education project funded and supported by the NGO ISA (Instituto Socioambiental, Brazil) and had the participation of 55 indigenous teachers from these 15 ethnics groups. The courses happened during 6 years, occurring two times a year and each lasting one month. The research data is from my field notes, taken when I was teaching the courses and in the villages visiting the schools and the communities, and also from interviews with indigenous teachers.

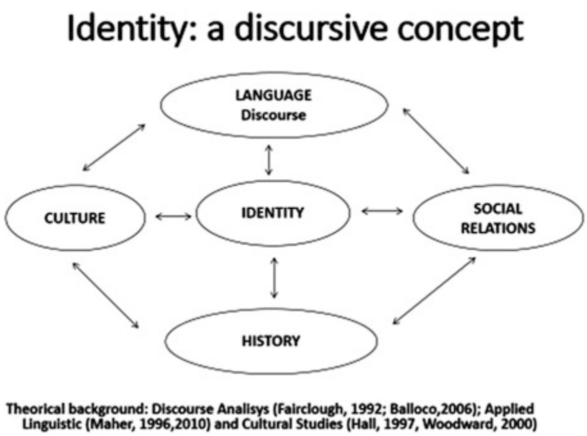
During their courses, the indigenous teachers, in a collaborative work, made a mathematics textbook to be used in teaching mathematics in their schools. When I proposed this task, I asked them if they would want to make a common book for all indigenous groups, written in Portuguese, or if each ethnic group would have their own book, written in their own indigenous language. Faced with this question, they chose to write it in their indigenous languages; they wanted to have their own book. In the process of producing the book, the Kaiabi teachers showed an initial problem relating to writing a mathematics textbook in their indigenous language. They would have problems with the ‘numbers’, because the Kaiabi language has numerical terms only up to five; nevertheless, they wanted a Kaiabi mathematics textbook.

It’s important to say that the written materials in indigenous language, in many indigenous teachers’ education programs, produced a movement of identity politics, in which the book written in a native language has become a marker of indigenous identity. Thereby, the questions of inquiry were: How do identity issues emerge from multilingual educational contexts? How do identity issues emerge from this context of indigenous teachers’ education? How is identity linked to the question of knowledge, in particular mathematical knowledge?

First, it is necessary to explain that the idea of indigenous identity in this work is taken from a discursive perspective, that is, identity is seen as a construct that happens in and through language—not in a fixed or essentialist sense, but in constant motion, because the identity processes are always built in relation to and dependent upon the nature of social relations that are established over time (see Fig. 1, from Fairclough, 1992; Hall, 1997; Maher, 1996, 2010; Woodward, 1997).

Balloco (2006) points out that thinking about the subject involves thinking about the constitution of social relations in its plurality. This is because the social relations occur in different times and different cultures, which causes a heterogeneous constitution of subjects throughout history, with different social formations. This plurality of social practices, Balloco underscores, is associated with a plurality of

Fig. 1 Identity concept



discursive positions. Thus, the identity constructs that constitute the subject historically occur from a variety of discursive representations. It is from this point of view that indigenous identity is discussed by Maher (1996). The author presents various facets of indigenous teachers' identities, and how indigenous language, in its oral and written forms, has been a major factor in the reconstruction of these identities. Concerning writing in indigenous language, the author points out the role it has taken in the assertion of ethnic identities as in the formation of other identities: Indian teacher historian and Indian teacher researcher. Thus, to Maher, discourse plays a key role in asserting distinguishing indigenous identities.

The following discussion focused on the Kaiabi teachers' discursive positioning of numerical terms in indigenous language, presented below, is underpinned by this theory that recognizes the centrality of language in identity.

2 Saying "Number" in Their Native Language: Aspects of Kaiabi Identity from Xingu Park

The indigenous group Kaiabi has lived in the Park since the 1960s, when they moved from their traditional lands because of constant conflicts and massacres they faced from invading rubber extractors. The invasion on extractive and agricultural fronts and the successive "peace" initiatives undertaken by the government agency for Indian affairs (SPI-Serviço de Proteção aos Indios), as well as the mission of Prelazia Diamantina led to the retreat and concentration of the population remaining from the fights and massacres in the area between the Peixe and Teles Pires rivers. Among this population, various Kaiabi families moved to the Indigenous Xingu Park between the years 1955 and 1966 to escape the plight of penury and persecution in which they found themselves (Franchetto, 1987).

The Kaiabi people have stood out in the ways they've sought to assert their identity as distinct from other groups in the Xingu. For the Kaiabi it is important to distinguish themselves from the surrounding society and other people of the park. In large part, the discursive practices present in the preparation of the mathematics textbook and in the interviews with Kaiabi teachers were related to the question of their ethnic identity, which appeared linked to the need to mark an identity of being Indian, as opposed to non-Indian and, especially, being a Kaiabi Indian, in contrast to other peoples of the Park. This occurred in relation to the numbers in indigenous language, through which the Kaiabi teachers marked an identity and a political position against the dominant decimal system. For this discussion it is important to understand the meanings involved in 'number' in both the Kaiabi culture and practices arising from contact with non-Indian society.

Originally, the Kaiabi's terminology for numbers in their native language only went up to five. To mark larger amounts, for example, the number of days of subsistence hunters spent in the woods, fingers and toes were used. For stays of more than twenty days, marking was done by the moon, which for Kaiabi has three phases: *Jay ruwi* (new moon), *Owauramu* (full moon) and *Ikuejowamu* (when the moon appears on both sides). Currently, Kaiabi teachers are writing numerical terms as follows: 1—***ajepeitee***; 2—***mukui***; 3—***muapyt***; 4—***irupawe***; 5—***irupawe irue’em***.

Although there is no name for larger numbers, this does not imply that the Kaiabi do not use numbers greater than five. The difference is related to the objective that the number will take on in the context of its use, either as a time marker, as in the example cited above, or to produce drawings of weaving, as I will explain below.

The Kaiabi produce artisanal weaving with standard baskets of woven designs (***yrupem***) that follow an order of learning, which are about 15 patterns (Fig. 2). These drawings that appear in the twisted baskets refer to the mythic narratives of the group (Ribeiro, 1986; Senra, 1996). In order to learn the design patterns, the first to be learned is named ***I’yp***, which the Kaiabi people call basket number one (Fig. 3).

When I was in the community following the research work done by an indigenous teacher on Kaiabi baskets, I asked about the initial formation of twisted baskets. In relation to basket number one, I asked how does one know the exact number of strips on the top of the braid. The teacher showed the drawing and said that this form points out the way to make the design. Afterwards, I asked about the number of strips in a vertical position. Instantly, the Kaiabi teacher counted one by one and said “seventeen”. Then I asked: “How? If the Kaiabi has numbers just to five?” Very quietly, the teacher replied that from the strip center (symmetrically), he counted on each side: three, three and two (Fig. 4).

Indeed, even though Kaiabi numerical terms do not go beyond five, there is a way of organizing the quantities that can be done by sorting groups of three, four, or two. In this context, the practices are related to the Kaiabi quantification, certainly not related to the idea of counting towards the accumulation of large quantities. The count has another purpose. In the case of weaving, it is expected to produce a

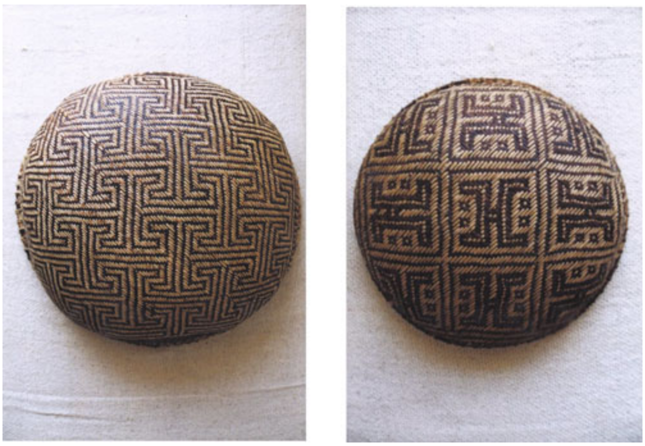


Fig. 2 Kaiabi baskets

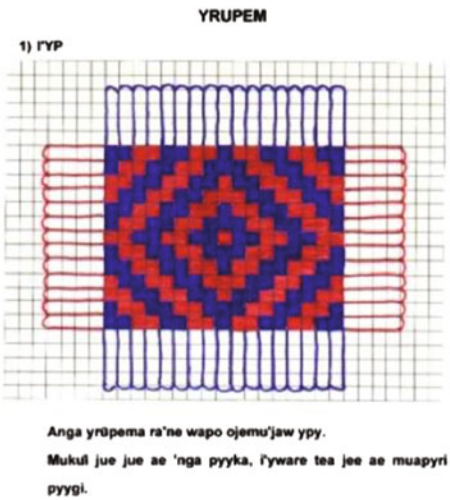


Fig. 3 Task in mathematics textbook about the Kaiabi basket number one

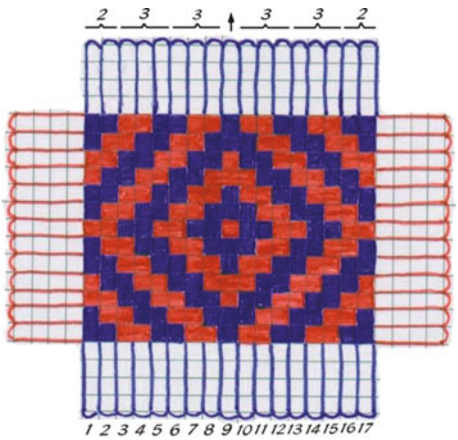


Fig. 4 Indigenous teacher explanation, in Portuguese, about the beginning of the basket number 1

particular design and for this the use of clusters is sufficient, without the need of a verbal description for the total number of strips. With social contact with non-indigenous society, mainly through trade in purchase and sale, came the new practices that now require the use of terms that designate larger numbers. The goal of the number appears within these new practices with the need to enumerate larger amounts and is also associated with the concept of monetary value.

Then, a question arises relating to the use of indigenous language and the Portuguese language in such practices. For this, some new terms were created in the native language. For example, for regarding money these terms are used: ***Ka'aranuu*** = money (which means any paper); ***Owuuma'e*** (large note, or large leaf) used for bills of denominations fifty and one hundred; ***Owiima'e*** (small note or small leaf) used for bills of ten and one; ***Y'wype'i*** (coin or round pods).

To give new meaning to the numerical magnitudes introduced by social contact, new terms were used in indigenous language, demonstrating which are the highest and lowest in terms of monetary value.

Changes in social activities have brought the Kaiabi into contact with another numeric universe, which was incorporated into their practices. That is because business practices have become part of the Kaiabi's life and cannot be seen exclusively as a non-indigenous activity. These new practices in this new numerical universe are already carried out within the Kaiabi context. However, the use of Portuguese for larger numbers created an obstacle, symbolic and political in nature, in producing the mathematics textbook on indigenous language. When the proposal to develop a mathematics textbook was presented to indigenous teachers, at first the Kaiabi immediately showed interest in having such material. All groups participating in the course opted to write a book on their indigenous languages (fifteen languages), including the Kaiabi. However, they were the only group that showed an initial difficulty, they would make a mathematics textbook written in indigenous language, but what would they do with the numbers? In their words "….How would we do the book if we only had numerical terms up to five?"

Nevertheless, some Kaiabi teachers claimed to have "*a math*", one said that they had "*too much math*" when they would build a house, but they "*found it hard*". They did not know how to put it in the book. Another Kaiabi teacher said: "*We want an indigenous Kaiabi math book. We want a book for working with students with 'Indian mathematical language'... how do we put in the book how to do Indian mathematics, because we have a lot of math when someone is going to build a home, a net, and an arrow.*"

We can recognize the presence of a discourse in defense of "an indigenous mathematics" in the Kaiabi teacher's remarks above, which is related to a "mathematical language." However, the issue of the number words originated by the contact with the dominant culture leads to this discourse defending and maintaining this "mathematical language." In accordance with this expectation, Kaiabi teachers then decided to hold a meeting with leaders and elders to discuss a proposed expansion of the numerical terminology in their indigenous language. After discussion among themselves, an Indian Kaiabi turned to me and said with a smile: "*Now we can say any number, one hundred, two hundred, three hundred....*". They approved a numerical terminology expansion up to ten.

The expansion of numerical terminology proposed by the Kaiabi, while it represents a result of the imposition of the dominant system, reveals characteristics of this community's sense of number. One is to rely on groups of three, four or two. The linguistic structure of the proposed terms, later six, follow the idea of organizing groups: 1—***ajepeitee***; 2—***mukui***; 3—***muapyt;*** 4—***irupawe*** (*iru* = pair; *pawe* = double); 5—***irupawe irue 'em*** (two pairs and one who does not have companion); 6—***muapyriru*** (three pairs); 7—***muapyrirue'em*** (three pairs and one who does not have companion); 8—***irupawepawe*** (four and four pairs), 9—***muapyjuejue*** (*muapy* is repeated three times), 10—***ae po jawe*** (it's the same number of all fingers in our hands).

The emphasis on the idea of numerical terms approved at the meeting of the elders, has a symbolic character. What is being reinforced in the terminology expansion is very much a matter of affirming the group identity. The ability to use greater numbers in the Kaiabi language implies that the group is not placed in an inferior position versus non-indigenous mathematics, nor versus other indigenous people, reinforced by the speech: "*Now we can say any number, one hundred, two hundred, three hundred.*"

Although it appears to be a contradictory discourse, because the extension was made based on the dominant system, base ten, the Kaiabi teachers appropriated this system when they created a new way of naming these numbers. Even in following a decimal system, the proposal introduces the notion of groups in the number sense understood by that community. For example, one teacher suggested that the Kaiabi number sixteen could be said in indigenous language ***ajeipeei muapyriru*** (one (ten) and three pairs), the reading appears in digits but retains the Kaiabi's sense of number, i.e., the six are seen as three groups of two.

The numerical concept of the decimal system first came with social contact. However, it became part of the Kaiabi's daily life. In an interview with a Kaiabi teacher, he said that when they were talking in indigenous language and they needed to say larger numbers, they borrowed from the Portuguese. This question arose in an interview with a Kaiabi teacher:

[1]Tarupi It is important that we increase the number for our counting, understand? Because only in counting are we not Indian and that isn't good, you know. So we have to increase our number… eh… to go higher because our counting is very low.

Jackie And why do you think that you need to speak with larger numbers in Kaiabi indigenous language?

Tarupi Ah, so… we need higher numbers to speak with our numbers with our relatives, we have to speak the same language.

Jackie With relatives.

Tarupi Yes, with relatives, but today we speak in indigenous language about everything, but when we have to speak about numbers, the numbers we speak in Portuguese.

Jackie That's, when you're talking to a relative, when you have to say the number you….

Tarupi The number speech is the white man's.

Jackie You speak in Portuguese.

Tarupi This is not good, you know. Now, if we have a number we could speak in our language and arrive at the time of counting, just speaking the number only in our language—this is important to us, you know. But if we only depend on the white man's language, we never get our own high numbers, you know.

[1]Translated from a variety of Portuguese, spoken by indigenous people, by the author.

To borrow a "non-indigenous speech" to say the Kaiabi numbers means that to speak a non-indigenous language is the same as perpetuating a dependent relationship: "*if we depend only on the white man's* (non-indigenous) *language, we never get our own high numbers*". The construction of new terms for these numbers in the native language is part of a movement of affirming the Kaiabi's identity. The number is no longer "*white man's* (non-indigenous) *speech*" and becomes "indigenous speech", or the "Kaiabi's speech or Kaiabi language".

3 Final Remarks

Identity processes are always in movement and, in intercultural encounter, marked by tension and the movement of coming and going (Bhabha, 1994). These processes generate a reinvention of identity sustained by relations between language and knowledge. The ways of making sense of knowledge in cultural practices by the subjects are intrinsically linked with identity as a discursive process. This discussion about how a 'number word' in indigenous language can be a defense of ethnic identity is seen as a movement of identity reinvention that depends on the nature of the social relationships which have been established over time. In the case of the Kaiabi people, speaking Portuguese is a historical and a political need. The intercultural contact is a given. Thus, they have to deal with this identity reinvention process all the time. The subjects, who emerge from a process of intercultural contact are engaged in identity policies. In this case it is possible to identify a reinvention of Kaiabi identity based on elements related to knowledge, in particular mathematical knowledge, and cultural practices that are produced and reconstructed by intercultural contact.

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© Springer International Publishing AG, part of Springer Nature 2018

J. N. Moschkovich et al. (eds.), *Language and Communication in Mathematics Education*, ICME-13 Monographs, https://doi.org/10.1007/978-3-319-75055-2

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